Transformations and Congruence

ESSENTIAL QUESTION
How can you use transformations and congruence to solve real-world problems?

Real-World Video
When a marching band lines up and marches across the field, they are modeling a translation. As they march, they maintain size and orientation. A translation is one type of transformation.

LESSON 9.1
Properties of Translations
CA CC 8.G.1, 8.G.3

LESSON 9.2
Properties of Reflections
CA CC 8.G.1, 8.G.3

LESSON 9.3
Properties of Rotations
CA CC 8.G.1, 8.G.3

LESSON 9.4
Algebraic Representations of Transformations
CA CC 8.G.3

LESSON 9.5
Congruent Figures
CA CC 8.G.2

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Get immediate feedback and help as you work through practice sets.
Are You Ready?

Assess Readiness
Use the assessment on this page to determine if students need intensive or strategic intervention for the module’s prerequisite skills.

Response to Intervention

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<td>Access Are You Ready? assessment online, and receive instant scoring, feedback, and customized intervention or enrichment.</td>
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Online and Print Resources

Skills Intervention worksheets
- Skill 47 Integer Operations
- Skill 89 Measure Angles

Differentiated Instruction
- Challenge worksheets
- PRE-AP

Extend the Math PRE-AP Lesson Activities in TE

Real-World Video Viewing Guide
After students have watched the video, discuss the following:
- What are some ways mentioned in the video that transformations are used in the real world?
- How do you move the band formation by using a transformation?
  Move each person the same number of steps up or down and left or right.

PROFESSIONAL DEVELOPMENT VIDEO

Author Juli Dixon models successful teaching practices as she explores the concept of real numbers in an actual eighth-grade classroom.

Online Teacher Edition
Access a full suite of teaching resources online—plan, present, and manage classes and assignments.

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Interactive Whiteboards
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Personal Math Trainer: Online Assessment and Intervention
Assign automatically graded homework, quizzes, tests, and intervention activities. Prepare your students with updated practice tests aligned with Common Core.
Before

Students understand:
• how to classify and draw plane figures
• how to graph plane figures on the coordinate plane
• congruence

In this module

Students use transformational geometry to represent:
• properties of orientation and congruence of translations in a coordinate plane
• properties of orientation and congruence of reflections in a coordinate plane
• properties of orientation and congruence of rotations in a coordinate plane
• the effect of translations, reflections, and rotations in a coordinate plane using an algebraic representation

After

Students will connect:
• transformations and congruence
• reflections over an axis and symmetry
• algebra and coordinate geometry
Use the examples on the page to help students know exactly what they are expected to learn in this module.

**CA Common Core Standards**

**Content Areas**

- Geometry—8.G

**Cluster**

Understand congruence and similarity using physical models, transparencies, or geometry software.

**8.G.2** Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

**8.G.3** Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

**8.G.1** Verify experimentally the properties of rotations, reflections, and translations:
- a Lines are taken to lines, and line segments to line segments of the same length.
- b Angles are taken to angles of the same measure.
- c Parallel lines are taken to parallel lines.

**Example 8.G.2**

The figure shows triangle \(ABC\) and its image after three different transformations. Identify and describe the translation, the reflection, and the rotation of triangle \(ABC\).

Figure 1 is a translation 4 units down. Figure 2 is a reflection across the \(y\)-axis. Figure 3 is a rotation of 180°.

**Example 8.G.3**

Rectangle \(RSTU\) with vertices \((-4, 1), (-1, 1), (-1, -3),\) and \((-4, -3)\) is reflected across the \(y\)-axis. Find the coordinates of the image.

The rule to reflect across the \(y\)-axis is to change the sign of the \(x\)-coordinate.

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<th>Reflect across the (y)-axis ((-x, y))</th>
<th>Coordinates of image</th>
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<td>((-4, 1), (-1, 1),) ((-1, -3), (-4, -3))</td>
<td>((-4, 1), (-1, 1),) ((-1, -3), (-4, -3))</td>
<td>((4, 1), (1, 1),) ((1, -3), (4, -3))</td>
</tr>
</tbody>
</table>

The coordinates of the image are \((4, 1), (1, 1), (1, -3),\) and \((4, -3)\).
Lesson Support

**Content Objective**  Students will learn how to describe the properties of translations and their effect on the congruence and orientation of figures.

**Language Objective**  Students will describe the properties of translation and their effect on the congruence and orientation of figures.

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**California Common Core Standards**

- **8.G.1** Verify experimentally the properties of rotations, reflections, and translations.
  a. Lines are taken to lines, and line segments to line segments of the same length.
  b. Angles are taken to angles of the same measure.
  c. Parallel lines are taken to parallel lines.

- **8.G.3** Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

- **MP.6** Attend to precision.

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**Building Background**

**Eliciting Prior Knowledge**  Review what it means for two polygons to have corresponding sides and corresponding angles. Then ask students to create a definition and example chart for congruent figures. Transition into the lesson by asking whether two congruent figures remain congruent if one of the figures is moved up or down or to the right or left.

**Cluster Connections**

This lesson provides an excellent opportunity to connect ideas in the cluster: *Understand congruence and similarity using physical models, transparencies, or geometry software*. Tell students that line segment \( AB \) has endpoints \( A(2, 1) \) and \( B(5, 3) \). Ask them to describe how each translation changes the coordinates of the endpoints of the preimage to create the coordinates of the endpoints of the image:

1. Translate \( AB \) 4 units to the right;
2. Translate \( AB \) 4 units to the left;
3. Translate \( AB \) 4 units up;
4. Translate \( AB \) 4 units down

Sample answer: (1) add 4 to the \( x \)-coordinate; \( y \)-coordinate stays same; (2) subtract 4 from the \( x \)-coordinate; \( y \)-coordinate stays same; (3) \( x \)-coordinate stays same; add 4 to the \( y \)-coordinate; (4) \( x \)-coordinate stays same; subtract 4 from the \( y \)-coordinate.
This lesson on properties of translations relies on students' understanding of the meaning of several words as they are used in mathematics: transformation, preimage, image, and translation. While English learners at the expanding and bridging levels of English proficiency might have encountered these words in other classes, they may be new to them in mathematics. Even with the definition in the lesson, students may benefit from adding an illustration along with the word to their word journals.

Building Background

pre – The term preimage is introduced in this lesson. Pre- is a prefix meaning before, earlier, or in front of. Other useful words beginning with the prefix pre- are precook, predict, and precaution.

tion – The terms transformation and translation are also introduced in this lesson. The suffix -tion means action. Other words ending in -tion are competition, exploration, and organization.

Leveled Strategies for English Learners

Emerging Have students at this level of English proficiency work in pairs to review and copy on graph paper the translations in Explore Activities 1 & 2. If possible, have them discuss in their primary language the steps to take to accomplish the translations.

Expanding Have students at this level of English proficiency work in pairs to redraw the translations in Explore Activities 1 & 2. Then have them list the steps they took to accomplish the translations.

Bridging Have students at this level of English proficiency work in pairs to redraw the translations in Explore Activities 1 & 2. Then have them describe for each other the steps they took to accomplish these translations.

Write out and model for students a sentence frame to begin their answer.

Yes, the figures are congruent because __________.
ESSENTIAL QUESTION
How do you describe the properties of translation and their effect on the congruence and orientation of figures? Sample answer: Translations preserve size, shape, and orientation.

Motivate the Lesson
Ask: What changes when you slide an object, such as a book, from one corner of your desk to different corners of your desk? Does the size or shape of the object change? Begin the Explore Activity to find out how to describe this action mathematically.

Explore
EXPLORE ACTIVITY 1
Focus on Modeling  Mathematical Practices
Ask students to move the triangle from the image position back to the preimage position and describe the movement. 7 units left and 5 units up. How does the description of the movement change? How does the description stay the same? Students should see that the magnitude of the movement stays the same, but the direction changes.

Explain
EXPLORE ACTIVITY 2
Connect Vocabulary  EL
Emphasize that a transformation is a function that describes a change in the position, size, or shape of a figure, and a translation is a type of transformation in which a shape changes position but not size or orientation. Students often mix up these two terms.

Questioning Strategies  Mathematical Practices
• How many different ways could trapezoid TRAP be translated? Justify your answer. It can be translated an infinite number of ways. The rule for the translation would be different for each way.
• What characteristics do you look for in an image to know that it has been translated from a preimage? The corresponding sides have the same lengths, and the corresponding angles have the same measures. The shape and size of the image is the same as that of the preimage. The orientation of the image and preimage are the same.

Engage with the Whiteboard
You may wish to have students measure the actual side lengths of the projected image and preimage on the whiteboard, or you can have students count the lengths of the sides in grid units (using the Distance Formula for PT and P’T’). Point out to students that although the lengths of the sides in centimeters on the projected image will not match the lengths in their books, the angle measurements will be the same.
**EXPLORE ACTIVITY 1**

**Exploring Translations**

You learned that a transformation is a function that assigns exactly one output to each input. A translation is a function that describes a change in the position, size, or shape of a figure. The output of a transformation is the image, and the input is the preimage.

A translation is a transformation that slides a figure along a straight line.

The triangle shown on the grid is the preimage (input). The arrow shows the motion of a translation and how point A is translated to point A'.

A. Trace triangle ABC and line AA' onto a piece of paper.
B. Slide your triangle along the line to model the translation that maps point A to point A'.
C. The image of the translation is the triangle produced by the translation. Sketch the image of the translation.
D. The vertices of the image are labeled using prime notation. For example, the image of A is A'. Label the images of points B and C.
E. Describe the motion modeled by the translation.
Move 7 units right and 5 units down.
F. Check that the motion you described in part E is the same motion that maps point A onto A', point B onto B', and point C onto C.

**Reflect**

1. How is the orientation of the triangle affected by the translation? 
   It is not affected. The orientation stays the same.

**EXPLORE ACTIVITY 2**

**Properties of Translations**

Use trapezoid TRAP to investigate the properties of translations.

A. Trace the trapezoid onto a piece of paper. Cut out your traced trapezoid.
B. Place your trapezoid on top of the trapezoid in the figure. Then translate your trapezoid 5 units to the left and 3 units up. Sketch the image of the translation by tracing your trapezoid in this new location.
C. Use a ruler to measure the sides of trapezoid TRAP in centimeters.
   \[ TR = 1.3 \text{ cm}, \quad RA = 1.7 \text{ cm}, \quad AP = 1.7 \text{ cm}, \quad TP = 1.75 \text{ cm} \]
D. Use a ruler to measure the sides of trapezoid TRAP' in centimeters.
   \[ TR' = 1.3 \text{ cm}, \quad RA' = 1.7 \text{ cm}, \quad AP' = 1.7 \text{ cm}, \quad TP' = 1.75 \text{ cm} \]
E. What do you notice about the lengths of corresponding sides of the two figures?
   The lengths of corresponding sides are the same.
F. Use a protractor to measure the angles of trapezoid TRAP.
   \[ m\angle T = 104^\circ, \quad m\angle R = 90^\circ, \quad m\angle A = 90^\circ, \quad m\angle P = 76^\circ \]
G. Use a protractor to measure the angles of trapezoid TRAP'.
   \[ m\angle T' = 104^\circ, \quad m\angle R' = 90^\circ, \quad m\angle A' = 90^\circ, \quad m\angle P' = 76^\circ \]
H. What do you notice about the measures of corresponding angles of the two figures?
   The measures of corresponding angles are the same.
I. Which sides of trapezoid TRAP are parallel? How do you know?
   TR and AP; They both lie along horizontal grid lines.
   Which sides of trapezoid TRAP' are parallel? TR' and AP'
   What do you notice? The sides that were parallel in the preimage remain parallel in the image.

**PROFESSIONAL DEVELOPMENT**

**Integrate Mathematical Practices MP.6**

This lesson provides an opportunity to address this Mathematical Practices standard. It calls for students to communicate precisely. Students translate a figure on a coordinate grid following a given translation rule. Then, students measure the lengths of the sides and the degrees of the angles to show that the corresponding sides and angles are congruent. Finally, students make a conjecture about the preservation of the size and shape of a figure.

**Math Background**

In future geometry courses, students will learn special conditions that guarantee that two triangles are congruent. It is not necessary to verify that all three pairs of sides are congruent and all three pairs of angles are congruent. There are numerous "shortcuts," such as the Side-Side-Side (SSS) Congruence Postulate, which states that two triangles are congruent if the corresponding sides of one triangle are congruent to the corresponding sides of the other triangle. In other words, if the corresponding sides are congruent, the angles must also be congruent. However, having all three corresponding angles congruent does not guarantee that the corresponding sides are necessarily congruent.
ADDITIONAL EXAMPLE 1
The figure shows triangle PQR. Graph the image of the triangle after a translation of 7 units to the right and 2 units up.

EXAMPLE 1

Questioning Strategies  
Mathematical Practices
- What do you notice about the purple lines in the first grid that show the translation of each vertex? They are parallel to each other. They have the same length and slope.
- What do you notice about the corresponding sides of the preimage and the image in Step 4? The corresponding sides are parallel and have the same length and slope.

Engage with the Whiteboard
On the second grid, have a volunteer plot points and draw triangle X"Y"Z", which is the image of X'Y'Z' after a translation exactly half the units of the one given in the example.

YOUR TURN

Avoid Common Errors
Students can check that they have not miscounted units in the translation of any one point by checking that the image and preimage have the same size, shape, and orientation.

Talk About It
Check for Understanding
Ask: What translation rule would have moved side DC to coincide with the x-axis?
a translation of 1 unit up and any distance left or right

Elaborate

Talk About It
Summarize the Lesson
Ask: How do you know when a transformation is a translation? The image will have the same size, shape, and orientation as the preimage.

GUIDED PRACTICE

Engage with the Whiteboard
To help students visualize Exercises 3–4, have volunteers sketch the images and preimages on a coordinate grid. In Exercise 3, have the students assign letters to the vertices as well.

Avoid Common Errors
Exercise 3–5 Remind students that a translation is a type of transformation. A translation only causes a change in the position of the figure; everything else remains the same.
Exercise 5 Remind students that the image will have its vertices labeled with the same letters as the corresponding preimage vertices, plus the symbol ′.
Graphing Translations

To translate a figure in the coordinate plane, translate each of its vertices. Then connect the vertices to form the image.

**EXAMPLE 1**

The figure shows triangle $XYZ$. Graph the image of the triangle after a translation of 4 units to the right and 1 unit up.

**STEP 1** Translate point $X$.
Count right 4 units and up 1 unit plot point $X'$.

**STEP 2** Translate point $Y$.
Count right 4 units and up 1 unit plot point $Y'$.

**STEP 3** Translate point $Z$.
Count right 4 units and up 1 unit plot point $Z'$.

**STEP 4** Connect $X'$, $Y'$, and $Z'$ to form triangle $X'Y'Z'$.

Yes, the figures are congruent. Translations preserve size and shape.

**Reflect**

2. **Make a Conjecture** Use your results from parts 1(a), 1(b), and 1(c) to make a conjecture about translations.

Sample answer: Translations preserve the size and shape of a figure, as well as its orientation.

3. Two figures that have the same size and shape are called congruent. What can you say about translations and congruence?

A translation produces a figure that is congruent to the original figure.

**Guided Practice**

1. **Vocabulary** A **transformation** is a change in the position, size, or shape of a figure.

2. **Vocabulary** When you perform a transformation of a figure on the coordinate plane, the input of the transformation is called the **preimage** and the output of the transformation is called the **image**.

3. Joni translates a right triangle 2 units down and 4 units to the right. How does the orientation of the image of the triangle compare with the orientation of the preimage? (Explore Activity 1)

The orientation will be the same.

4. Rashid drew rectangle $PQRS$ on a coordinate plane. He then translated the rectangle 3 units up and 3 units to the left and labeled the image $PQRS'$. How do rectangle $PQRS$ and rectangle $PQRS'$ compare? (Explore Activity 2)

They are congruent.

5. The figure shows trapezoid $WXYZ$. Graph the image of the trapezoid after a translation of 4 units up and 2 units to the left. (Example 1)

6. What are the properties of translations?

Sample answer: Translations preserve the size, shape, and orientation of a figure.

**World History**

A zoetrope is a device that consists of a cylinder with slits cut vertically around its sides. On the inside of the cylinder is drawn a series of pictures of the same object translated to different positions. When the zoetrope is spun, a person looking through the slits sees what appears to be the object in motion. The earliest known zoetrope was created in China around 180 CE.

**Additional Resources**

**Differentiated Instruction** includes:
- Reading Strategies
- Success for English Learners
- Reteach
- Challenge

**Curriculum Integration**

Have groups of students make up a pattern of dance steps formed by translating shapes, which represent dancers’ feet, on a grid. Then have students show the class their “dance” using the tiles on the floor as an enlarged grid.
**Evaluate**

**GUIDED AND INDEPENDENT PRACTICE**

8.G.1, 8.G.1a, 8.G.1b, 8.G.1c, 8.G.3

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**Answers**

1. Graph triangle $ABC$ with vertices $A(-3, 4)$, $B(0, 2)$, and $C(-2, 1)$ on a coordinate grid.

1. Graph the image of triangle $ABC$ after a translation of 4 units right and 3 units down.
2. Which side of the image is congruent to side $AB$?
3. Which angle in the image is congruent to angle $B$?
4. Angle $G$ in quadrilateral $FGHJ$ measures $135^\circ$. Brent translates the quadrilateral 3 units right and 1 unit up. What is the measure of the image of angle $G$?

Lesson Quiz available online

**Additional Resources**

*Differentiated Instruction* includes:
- Leveled Practice worksheets
9.1 Independent Practice

7. The figure shows triangle DEF.
   a. Graph the image of the triangle after the translation that maps point D to point D'.
   b. How would you describe the translation?
      The translation moved the triangle 2 units to the left and 4 units down.
   c. How does the image of triangle DEF compare with the preimage?
      They are congruent.

8. a. Graph quadrilateral KLMN with vertices K(3, 2), L(2, 2), M(0, -3), and N(-4, 0) on the coordinate grid.
   b. On the same coordinate grid, graph the image of quadrilateral KLMN after a translation of 3 units to the right and 4 units up.
   c. Which side of the image is congruent to side LM?

   Draw the image of the figure after each translation.

9. 4 units left and 2 units down
10. 5 units right and 3 units up

11. The figure shows the ascent of a hot air balloon. How would you describe the translation?
    The hot air balloon was translated 4 units to the right and 5 units up.

12. Critical Thinking Is it possible that the orientation of a figure could change after it is translated? Explain.
    No; when a figure is translated, it is slid to a new location. Since it is not turned or flipped, the orientation will remain the same.

13. a. Multistep Graph triangle XYZ with vertices X(-2, -3), Y(-2, 2), and Z(4, -4) on the coordinate grid.
   b. On the same coordinate grid, graph and label triangle X'Y'Z', the image of triangle XYZ after a translation of 3 units to the left and 6 units up.
   c. Now graph and label triangle X''Y''Z'', the image of triangle X'Y'Z' after a translation of 1 unit to the left and 2 units down.
   d. Analyze Relationships How would you describe the translation that maps triangle XYZ onto triangle X''Y''Z''?
      Sample answer: The original triangle was translated 4 units up and 4 units to the left.

14. Critical Thinking The figure shows rectangle PQRST, the image of rectangle PQRS after a translation of 5 units to the right and 7 units up. Graph and label the preimage PQRS.

15. Communicate Mathematical Ideas Explain why the image of a figure after a translation is congruent to its preimage.
    Sample answer: Since every point of the original figure is translated the same number of units up/down and left/right, the image is exactly the same size and shape as the preimage. Only the location is different.

EXTEND THE MATH

Activity A strip pattern is a design that repeats itself along a straight line. Some strip patterns, like the one shown, are examples of translations. On a strip of paper, create a design. Then repeat the design by translating it along a straight line to create your own strip pattern.
Lesson Support

**Content Objective** Students will learn how to describe the properties of reflections and their effect on the congruence and orientation of figures.

**Language Objective** Students will explain how to describe the properties of reflection and their effect on the congruence and orientation of figures.

### California Common Core Standards

- **8.G.1** Verify experimentally the properties of rotations, reflections, and translations.
  - a. Lines are taken to lines, and line segments to line segments of the same length.
  - b. Angles are taken to angles of the same measure.
  - c. Parallel lines are taken to parallel lines.

- **8.G.3** Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

- **MP.5** Use appropriate tools strategically.

### Building Background

**Eliciting Prior Knowledge** Ask students to work with a partner or in small groups to create a concept map based on reflections. The map should include both real-life connections and whatever geometrical concepts students can recall about reflections.

### Learning Progressions

In this lesson, students verify experimentally the properties of reflections by reflecting polygons in the coordinate plane across the axes. Important understandings for students include the following:

- Investigate the properties of reflections.
- Graph reflections.

Students are familiar with reflections, or flips, from earlier grades. This lesson provides an examination of the properties of reflections by having students measure and compare lengths and angles of corresponding parts of preimages and images. Students apply their understanding by graphing reflections of polygons in the coordinate plane.

### Cluster Connections

This lesson provides an excellent opportunity to connect ideas in the cluster: **Understand congruence and similarity using physical models, transparencies, or geometry software**. Tell students that triangle ABC has vertices A(1, 1), B(3, 1), and C(2, 4). Ask them to describe how each reflection changes the coordinates of the vertices of the preimage to create the coordinates of the vertices of the image:

1. Across the x-axis
2. Across the y-axis

Sample answer: (1) the z-coordinates stay the same and the y-coordinates have opposite signs from the original; (2) the x-coordinates have opposite signs from the original and the y-coordinates stay the same.
Language Support

California ELD Standards

Emerging 2.I.1. Exchanging information/ideas — Contribute to conversations and express ideas by asking and answering yes-no and wh-questions and responding using short phrases.

Expanding 2.I.1. Exchanging information/ideas — Contribute to class, group, and partner discussions, including sustained dialogue, by following turn-taking rules, asking relevant questions, affirming others, and adding relevant information.

Bridging 2.I.1. Exchanging information/ideas — Contribute to class, group, and partner discussions, including sustained dialogue, by following turn-taking rules, adding relevant information, building on responses, and providing useful feedback.

Linguistic Support

Academic/Content Vocabulary
This lesson relies on students’ understanding of the meaning of reflection and line of reflection. Point out that these terms are defined in the context of the sentence in which they are introduced. Also, have students turn to the glossary to see the new word defined. Often there is a visual diagram or other support, including a Spanish language explanation, in the glossary. For a glossary resource in 13 world languages, be sure to take students to the Online Multilingual Glossary.

Building Background
Help English learners figure out the meaning of unknown words in the lesson by pointing out patterns in words. In Explore Activity 1 of this lesson, students are instructed to fold and unfold a piece of paper. The prefix un- means not. So, unfold means not fold. Other examples of words that begin with this prefix are able/unable, do/undo, fair/unfair, and decided/undecided. Suggest they add this information to their math journals.

Leveled Strategies for English Learners

Emerging Have students at this level of English proficiency work in pairs to illustrate and label on graph paper a translation and reflection of the same size and shape. Ask them to show or point to how they are similar and how they are different.

Expanding Have students at this level of English proficiency work in pairs to define and illustrate on graph paper a translation and reflection of the same size and shape. Have students write down how the two are different.

Bridging Have students at this level of English proficiency work in pairs to discuss, illustrate, and label the differences between translation and reflection of the same size and shape.

Math Talk

A reflection produces a figure that is congruent ________.


**Engage**

**ESSENTIAL QUESTION**

*How do you describe the properties of reflection and their effect on the congruence and orientation of figures?*  
Sample answer: Reflections preserve size and shape, but not orientation.

**Motivate the Lesson**

**Ask:** What changes when you flip an object, such as a book, in any direction? Does the size or shape of the object change? Begin the Explore Activity to find out how to describe this action mathematically.

**Explore**

**EXPLORE ACTIVITY 1**

**Focus on Modeling**

After students have folded their paper across the axes and drawn both reflections, have them fold their papers over both axes at the same time, folding the paper into quarters. If they have drawn the reflections correctly, all three figures should match up exactly.

**Mathematical Practices**

**MP.5 Using Tools**

**EXPLORE ACTIVITY 2**

**Connect Vocabulary**

Point out that translations and reflections are both types of transformations. While a translation does not change the orientation of a figure, a reflection does. Emphasize that a *line of reflection* is often one of the axes, but it can be any line, including lines that are not horizontal or vertical.

**Questioning Strategies**

- How many different ways could trapezoid TRAP be reflected? Justify your answer. It can be reflected an infinite number of ways. Any line can be a line of reflection, and there is an infinite number of lines.
- What characteristics do you look for in an image to know that it has been reflected from a preimage? The corresponding sides have the same lengths, and the corresponding angles have the same measures. The shape and size of the image is the same as that of the preimage. The only difference is the image is a mirror image of the preimage.

**Engage with the Whiteboard**

Have students draw the reflection of trapezoid TRAP across the x-axis. Name the new image TR’A’P’. Students can also draw the reflection of TR’AP across the x-axis and name the new image TR’’A’’P’’.
Properties of Reflections

**EXPLORE ACTIVITY 1**  
Exploring Reflections

A reflection is a transformation that flips a figure across a line. The line is called the line of reflection. Each point and its image are the same distance from the line of reflection.

The triangle shown on the grid is the preimage. You will explore reflections across the x- and y-axes.

- **A** Trace triangle ABC and the x- and y-axes onto a piece of paper.
- **B** Fold your paper along the x-axis and trace the image of the triangle on the opposite side of the axis. Unfold your paper and label the vertices of the image A', B', and C'.
- **C** What is the line of reflection for this transformation?
- **D** Find the perpendicular distance from each point to the line of reflection.
  - Point A: 5 units
  - Point B: 2 units
  - Point C: 2 units
- **E** Find the perpendicular distance from each point to the line of reflection.
  - Point A': 5 units
  - Point B': 2 units
  - Point C': 2 units
- **F** What do you notice about the distances you found in **D** and **E**?
  - They are the same for a point and its reflection.

**Reflect**

1. Fold your paper from **A** along the y-axis and trace the image of triangle ABC on the opposite side. Label the vertices of the image A', B', and C'. What is the line of reflection for this transformation?

2. How does each image in your drawings compare with its preimage?

Sample answer: \( \triangle A'B'C' \) is \( \triangle ABC \) flipped across the x-axis, \( \triangle A''B''C'' \) is \( \triangle ABC \) flipped across the y-axis.

**EXPLORE ACTIVITY 2**

Properties of Reflections

Use trapezoid TRAP to investigate the properties of reflections.

- **A** Trace the trapezoid onto a piece of paper. Cut out your traced trapezoid.
- **B** Place your trapezoid on top of the trapezoid in its new location. Label the vertices of the image \( T', R', A', \) and \( P' \).
- **C** Use a ruler to measure the sides of trapezoid \( TRAP \) in centimeters.
  - \( TR = 0.9 \text{ cm} \)
  - \( RA = 0.8 \text{ cm} \)
  - \( AP = 1.2 \text{ cm} \)
  - \( TP = 2.1 \text{ cm} \)
- **D** Use a ruler to measure the sides of trapezoid \( T'R'A'P' \) in centimeters.
  - \( T'R = 0.9 \text{ cm} \)
  - \( R'A' = 0.8 \text{ cm} \)
  - \( A'P' = 1.2 \text{ cm} \)
  - \( P'T' = 2.1 \text{ cm} \)
- **E** What do you notice about the lengths of corresponding sides of the two figures?
  - The lengths of corresponding sides are the same.

- **F** Use a protractor to measure the angles of trapezoid \( TRAP \).
  - \( \angle T = 63^\circ \)
  - \( \angle R = 117^\circ \)
  - \( \angle A = 135^\circ \)
  - \( \angle P = 45^\circ \)
- **G** Use a protractor to measure the angles of trapezoid \( T'R'A'P' \).
  - \( \angle T' = 63^\circ \)
  - \( \angle R' = 117^\circ \)
  - \( \angle A' = 135^\circ \)
  - \( \angle P' = 45^\circ \)
- **H** What do you notice about the measures of corresponding angles of the two figures?
  - The measures of corresponding angles are the same.

- **I** Which sides of trapezoid \( TRAP \) are parallel? \( TP \) and \( RA \)
- **J** Which sides of trapezoid \( T'R'A'P' \) are parallel? \( T'P' \) and \( R'A' \)
- **K** What do you notice?
  - The sides that were parallel in the preimage remain parallel in the image.

**PROFESSIONAL DEVELOPMENT**

**Integrate Mathematical Practices MP.5**

This lesson provides an opportunity to address this Mathematical Practices standard. It calls for students to use tools such as models, rulers, and pencil and paper to analyze relationships. Students use the results of the Explore Activities to make a conjecture that reflections preserve the size and shape of a figure. They find the measures of the angles and side lengths of the image and its preimage and use them to justify their conjecture.

**Math Background**

Translations, reflections, and rotations are examples of rigid motions. Reflections are sometimes called improper rigid motions as they can cause shapes to flip into a new orientation. However, applying the same reflection twice results in the original orientation. Translations and rotations are sometimes called proper rigid motions as they never cause the shape to flip.
EXAMPLE 1

**Questioning Strategies**

- What do you notice about the purple lines that show the reflection of each vertex? They are parallel to each other. The line of reflection divides each line in half, but all three lines are not the same length.

- Why is triangle $XYZ'$ not a translation of triangle $XYZ$? The figures do not have the same orientation.

**Engage with the Whiteboard**

Have a volunteer plot a new triangle, $X'Y'Z'$, that is a reflection of $XYZ$ across the $y$-axis. Have students compare the orientations of the three triangles.

YOUR TURN

**Avoid Common Errors**

Students may plot the vertices of the image correctly but label them incorrectly. Suggest that they label each vertex of the image as they plot the point and confirm that each letter matches the letter of the corresponding vertex in the preimage.

**Talk About It**

Ask: How do the coordinates of the image differ from the coordinates of the preimage when a figure is reflected across the $y$-axis? The $x$-values are the opposite of the preimage's $x$-values, but the $y$-values remain the same.

Elaborate

**Talk About It**

**Summarize the Lesson**

Ask: How do you know when a transformation is a reflection? The image will have the same size and shape as the preimage, but the orientation will not be the same. There will be a line of reflection such that each image point will be the same distance from that line as its corresponding preimage point.

**GUIDED PRACTICE**

**Engage with the Whiteboard**

For Exercise 2 have students graph the reflections of trapezoid $ABCD$ across both the $x$-axis and the $y$-axis on the coordinate grid. Label the images $A'B'C'D'$ and $A''B''C''D''$.

**Avoid Common Errors**

**Exercise 2a** Remind students that the image will have its vertices labeled with the same letters as the corresponding preimage vertices, plus the symbol '.
Reflect

3. Make a Conjecture Use your results from 6, 10, and 14 to make a conjecture about reflections.

Sample answer: Reflections preserve the size and shape of a figure, but the orientation changes to a mirror image of the original.

Graphing Reflections

To reflect a figure across a line of reflection, reflect each of its vertices. Then connect the vertices to form the image. Remember that each point and its image are the same distance from the line of reflection.

**EXAMPLE 1**

The figure shows triangle XYZ. Graph the image of the triangle after a reflection across the x-axis.

**STEP 1** Reflect point X.
Point X is 3 units below the x-axis. Count 3 units above the x-axis and plot point X'.

**STEP 2** Reflect point Y.
Point Y is 1 unit below the x-axis. Count 1 unit above the x-axis and plot point Y'.

**STEP 3** Reflect point Z.
Point Z is 5 units below the x-axis. Count 5 units above the x-axis and plot point Z'.

**STEP 4** Connect X', Y', and Z' to form triangle X'Y'Z'.

Each vertex of the image is the same distance from the x-axis as the corresponding vertex in the original figure.

Math Talk: A reflection produces a figure that is congruent to the original figure.

**Guided Practice**

1. Vocabulary A reflection is a transformation that flips a figure across a line called the line of reflection.

2. The figure shows trapezoid ABCD. (Explore Activities 1 and 2 and Example 1)
   a. Graph the image of the trapezoid after a reflection across the x-axis. Label the vertices of the image.
   b. How do trapezoid ABCD and trapezoid A'B'C'D' compare?
      They are congruent.
   c. What if? Suppose you reflected trapezoid ABCD across the y-axis. How would the orientation of the image of the trapezoid compare with the orientation of the preimage?
      The orientation would be reversed horizontally. That is, the figure from left to right in the preimage would match the figure from right to left in the image.

3. What are the properties of reflections?
   Sample answer: Reflections preserve size and shape but not orientation.

**Differentiate Instruction**

**Cooperative Learning**

Provide each student with a full sheet of grid paper. Have each student fold their paper into quarters along grid lines. They should mark the fold lines as the x- and y-axes. Each student should draw half of a face or design to the left of the y-axis. The right edge of the face or design should touch the y-axis. Students then trade with another student to complete the face or design by drawing a reflection of the drawing across the y-axis.

**Critical Thinking**

Pose this question to your students: If a cat is sitting 8 inches away from the front of a mirror, how far away will the cat’s reflection appear to be for the cat? The reflection of the cat will appear to be twice the distance the cat is from the mirror, or 16 inches.

**Additional Resources**

*Differentiated Instruction* includes:
- Reading Strategies
- Success for English Learners (EL)
- Reteach
- Challenge (PRE-AP)
Evaluate

GUIDED AND INDEPENDENT PRACTICE

- **Explore Activity 1**
  - Exploring Reflections
  - Exercises 2, 4–5, 9

- **Explore Activity 2**
  - Properties of Reflections
  - Exercises 2, 4–5, 7–8

- **Example 1**
  - Graphing Reflections
  - Exercises 2, 8

**Concepts & Skills**

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<tr>
<td>10–11</td>
<td>3 Strategic Thinking</td>
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**Additional Resources**

- Leveled Practice worksheets

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**Answers**

1. Graph the image of triangle $ABC$ after a reflection across the $y$-axis.

2. Which side of the image is congruent to side $\overline{AB}$?

3. Which angle in the image is congruent to angle $B$?

4. If a point $M$, 5 units from the $x$-axis, is reflected across the $x$-axis, how far is the image of the point, $M'$, from the $x$-axis?

5. Angle $G$ in trapezoid $FGHJ$ measures $135^\circ$. Jasmine reflects the trapezoid over the $x$-axis. What is the measure of the image of angle $G$?

Lesson Quiz available online

**Answers**

1. Graph the image of triangle $ABC$ after a reflection across the $y$-axis.

2. $\overline{A'B'}$

3. angle $B'$

4. 5 units

5. $135^\circ$
9.2 Independent Practice

The graph shows four right triangles. Use the graph for Exercises 4–7.

8. a. Graph quadrilateral WXYZ with vertices W(−2, −2), X(3, 3), Y(5, −1), and Z(4, −6) on the coordinate grid.

b. On the same coordinate grid, graph quadrilateral WXYZ, the image of quadrilateral WXYZ, after a reflection across the x-axis.

c. Which side of the image is congruent to side YZ? Name three other pairs of congruent sides.

WX and W'X', XY and X'Y', WZ and W'Z'.

d. Which angle of the image is congruent to ∠X? Name three other pairs of congruent angles.

∠W and ∠W', ∠Y and ∠Y', ∠Z and ∠Z'.

Activity

Reflect triangle ABC across line ℓ. Label the image triangle A'B'C'. Then reflect triangle A'B'C' across line m. Label the image triangle A''B''C''. What other transformation could you have performed on triangle ABC to get triangle A'B'C'? Do you think this would be true of any shape that goes through the same process?

Sample answer: A translation of triangle ABC could have produced A'B'C'. Yes, any shape would be reversed after one reflection, then reversed again to the original figure after the second reflection only if the lines of reflection are parallel.
Lesson Support

**Content Objective**  Students will learn how to describe the properties of rotations and their effects on the congruence and orientation of figures.

**Language Objective**  Students will describe the properties of rotations and their effects on the congruence and orientation of figures.

### California Common Core Standards

<table>
<thead>
<tr>
<th>Code</th>
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| 8.G.1 | Verify experimentally the properties of rotations, reflections, and translations.  
| a. | Lines are taken to lines, and line segments to line segments of the same length.  
| b. | Angles are taken to angles of the same measure.  
| c. | Parallel lines are taken to parallel lines. |
| 8.G.3 | Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates. |
| MP.2 | Reason abstractly and quantitatively. |

### Building Background

**Connecting to Everyday Life**  Ask students to explain what a rotation means in their own words. Then have them describe examples of rotations with which they are familiar. Possible examples include an analog clock, gears, wind and water mills, Ferris wheels, tires, planetary orbits, and Earth’s rotation. Elicit the observation that all of the examples rotate about a center point, the center of rotation.

### Learning Progressions

In this lesson, students verify experimentally the properties of reflections by rotating polygons in the coordinate plane around a given center of rotation. Important understandings for students include the following:

- Investigate the properties of rotations.
- Graph rotations.

Like the other rigid transformations, students learn that a rotation preserves congruence and like reflections can change orientation. They rotate figures 90°, 180°, and 270°, clockwise and counterclockwise. Students wind up their review and practice with rigid transformations in this lesson in preparation for analyzing these transformations as functions algebraically in the next lesson.

### Cluster Connections

This lesson provides an excellent opportunity to connect ideas in the cluster: **Understand congruence and similarity using physical models, transparencies, or geometry software.** The center of rotation of a polygon can be inside the polygon. Have students graph an octagon with vertices A(1, 3), B(1, 1), C(2, 0), D(2, −2), E(−2, −2), F(−2, 0), G(−1, 1), and H(−1, 3). Ask students to rotate the octagon about the origin clockwise 90°, 180°, and 270°. Have them record the coordinates of preimage vertex A and the corresponding rotation images of A′, A″, and A‴.

A(2, 3), A′(3, −2), A″(−2, −3), A‴(−3, 2)
This lesson relies on students’ understanding of the meanings of \textit{rotation} and \textit{center of rotation}. These terms are defined in the context of the sentence in which they are introduced. Have students also turn to the glossary to see the new word defined as well as a visual diagram. For a glossary resource for 13 world languages, including Spanish, be sure to take students to the Online Multilingual Glossary.

The words \textit{rotate}, \textit{rotation}, \textit{reflect}, and \textit{reflections} are found throughout this lesson. Many English verbs can be turned into nouns by adding the suffix \textit{-tion} to them. Words that end in \textit{-tion} in English are often cognates in Spanish (\textit{rotación}, \textit{reflexión}).

The prefix \textit{counter-}

The prefix \textit{counter-} is also found in this lesson in the word \textit{counterclockwise}. Provide a demonstration of \textit{clockwise} first. Then point out that \textit{counter} is like \textit{contra}, it means \textit{against}.

When proficiency in English is limited, having students use their primary language in peer-to-peer discussion encourages higher-level thinking.

Have students at this level of English proficiency work in small groups of mixed language proficiency to list the properties of rotations.

To make sure that the nuances of language have not prevented students from understanding the concepts, have them separate the parts of the essential question to help them answer it.

Write out and model for students a sentence frame to begin their answer.

\textit{The orientation of the triangle is affected by \underline{________}.}
Engage

**ESSENTIAL QUESTION**

*How do you describe the properties of rotation and their effect on the congruence and orientation of figures?*

Sample answer: Rotations preserve size and shape, but change orientation.

Motivate the Lesson

**Ask:** What changes when you turn an object, such as a book, around a point? Does the size or shape of the object change? Begin the Explore Activity to find out how to describe this action mathematically.

Explore

**EXPLORE ACTIVITY 1**

**Mathematical Practices**

- **Focus on Modeling**
  - Make sure students understand that point $A$ is the same as point $A'$ because $A$ lies at the center of rotation. The next Explore Activity shows a rotation where none of the vertices lie at the center of rotation, and therefore all of the vertices change position.

**EXPLORE ACTIVITY 2**

**Connect Vocabulary**

- Emphasize that a transformation is a function that describes a change in the position, size, or shape of a figure, and a rotation is a type of transformation. The measures of the figure's sides and angles do not ever change in a rotation. In most rotations, the figure's position and orientation change.

**Questioning Strategies**

- If you were to draw segments from $T$ to the origin and from $T'$ to the origin, what angle would the two segments form? **They would form a 180° or straight angle.**
- How does the distance from the origin to $T$ compare to the distance from the origin to $T'$? **The distance is the same.**
- Would you give the same answers to the previous two questions for each of the other vertices and their images? **Yes, each pair of vertices (preimage and image) would form a 180° angle with the origin, and their distance from the origin would be the same.**

**Engage with the Whiteboard**

- Have students draw semicircular arrows, with the center of the semicircle at the origin, connecting $T$ with $T'$, $A$ with $A'$, $R$ with $R'$ and $P$ with $P'$. Point out that in this case, counterclockwise and clockwise arrows are equally valid.

**Focus on Critical Thinking**

- Point out to students that a clockwise rotation of 270° results in the same image as a counterclockwise rotation of 90°. Ask students to examine this claim, discuss why it is true, and justify it with a logical argument. **Sample answer:** Since $270° + 90° = 360°$, and a full rotation is 360°, then rotating 270° in one direction is the same as rotating 90° in the opposite direction.
**EXPLORE ACTIVITY 1**

**Exploring Rotations**

A rotation is a transformation that turns a figure around a given point called the center of rotation. The image has the same size and shape as the preimage.

The triangle shown on the grid is the preimage. You will use the origin as the center of rotation.

1. **Trace triangle ABC onto a piece of paper. Cut out your traced triangle.**
2. **Rotate your triangle 90° counterclockwise about the origin.**
3. **Sketch the image of the rotation.** Label the images of points A, B, and C as A', B', and C'.

**Reflect**

1. **Communicate Mathematical Ideas.** How are the size and the orientation of the triangle affected by the rotation?
   - The size stays the same, but the orientation changes in that the triangle is turned or tilted left – what was “up” is now “left.”
2. **Rotate triangle ABC 90° clockwise about the origin.** Sketch the result on the coordinate grid above. Label the image vertices A', B', and C'.

**PROFESSIONAL DEVELOPMENT**

**Integrate Mathematical Practices MP.2**

This lesson provides an opportunity to address this Mathematical Practices standard. It calls for students to make sense of relationships in a problem. Students use coordinate grids to visualize a relationship between a preimage and a rotation that results in an image. Then students use words to describe the relationship between the preimage and the image following a rotation.

**Math Background**

A rotation is a mathematical model of the motion of turning. It is a transformation. To rotate a figure you must be given or know three things: the center of rotation, the magnitude (number of degrees), and direction (clockwise or counterclockwise) of the rotation.

**EXPLORE ACTIVITY 2**

**Properties of Rotations**

Use trapezoid TRAP to investigate the properties of rotations.

1. **Trace the trapezoid onto a piece of paper. Include the portion of the x- and y-axes bordering the third quadrant. Cut out your tracing.**
2. **Place your trapezoid and axes on top of those in the figure.** Then use the axes to help rotate your trapezoid 180° counterclockwise about the origin. Sketch the image of the rotation of your trapezoid in this new location. Label the vertices of the image T', R', A', and P'.
3. **Use a ruler to measure the sides of trapezoid TRAP in centimeters.**
   - TR = 1.3 cm, RA = 1.7 cm
   - AP = 1.5 cm, TP = 2.2 cm
4. **Use a ruler to measure the sides of trapezoid TRAP' in centimeters.**
   - TR' = 1.3 cm, RA' = 1.7 cm
   - AP' = 1.5 cm, TP' = 2.2 cm
5. **What do you notice about the lengths of corresponding sides of the two figures?**
   - The lengths of corresponding sides are the same.
6. **Use a protractor to measure the angles of trapezoid TRAP.**
   - m∠T = 90°, m∠R = 90°, m∠A = 108°, m∠P = 72°
7. **Use a protractor to measure the angles of trapezoid TRAP'.**
   - m∠T = 90°, m∠R = 90°, m∠A' = 108°, m∠P' = 72°
8. **What do you notice about the measures of corresponding angles of the two figures?**
   - The measures of corresponding angles are the same.
9. **Which sides of trapezoid TRAP are parallel?** TR and RA
   - Which sides of trapezoid TRAP' are parallel? TR' and RA'
10. **What do you notice?** The sides that were parallel in the preimage remain parallel in the image.
ADDITIONAL EXAMPLE 1
The figure shows triangle PQR. Graph the image of the triangle after a clockwise rotation of 90° about the origin.

EXAMPLE 1
Questioning Strategies Mathematical Practices
- About what point do you rotate triangle ABC? point A.
- Triangle ABC is in the first and second quadrants. In which quadrants will the image lie? Quadrants I and IV.

Engage with the Whiteboard
Have a student draw triangle A'B'C', which is A'B'C' after a 90° clockwise rotation. Have another student draw A''B''C'' after another 90° clockwise rotation. Have students predict what another 90° rotation would produce.

YOUR TURN
Avoid Common Errors
Students often confuse clockwise and counterclockwise when performing a rotation. Draw or show the diagram below to show the meanings of the words.

Talk About It
Check for Understanding
Ask: Looking at your answer to Exercise 6, what indicates that quadrilateral ABCD was not translated to get quadrilateral A'B'C'D'? The size and shape of the figures are the same, but the orientation is different.

Elaborate
Talk About It
Summarize the Lesson

GUIDED PRACTICE
Engage with the Whiteboard
For Exercises 2–5, have volunteers circle the degree, direction, and center of rotation of the figures. In Exercises 4 and 5, draw curved arrows in the direction of rotation.

Avoid Common Errors
Exercise 5 Make sure students label the image correctly. Some students may swap B' and D' out of carelessness or thinking the order of the labels doesn't matter.
Reflect
3. Make a Conjecture Use your results from 1, 2, 4, and 5 to make a conjecture about rotations.
Sample answer: Rotations preserve size and shape, or congruence, but change a figure's orientation by turning it.

4. Place your tracing back in its original position. Then perform a 180° clockwise rotation about the origin. Compare the result with the result in 2.
A 180° clockwise rotation gives the same image as a 180° counterclockwise rotation.

Guided Practice
1. Vocabulary A rotation is a transformation that turns a figure around a given __________ point called the center of rotation.

Siobhan rotates a right triangle 90° counterclockwise about the origin.

2. How does the orientation of the image of the triangle compare with the orientation of the preimage? (Explore Activity 1)

Each leg in the preimage is perpendicular to its corresponding leg in the image.

3. Is the image of the triangle congruent to the preimage? (Explore Activity 2)

Yes, the figures are congruent.

Draw the image of the figure after the given rotation about the origin. (Example 1)

4. 90° counterclockwise

5. 180°

6. What are the properties of rotations?

Sample answer: Rotations preserve size and shape but change orientation.

Math on the Spot my.hrw.com

Math Talk
How does the orientation of the triangle affect the image affected by the rotation?

Sample answer: The triangle is turned to the right about the origin by the angle of rotation.

Differentiate Instruction

Modeling
Students who have trouble finding the location of an image after a rotation may benefit from finding the image of just one point after a rotation. Provide students with examples like the one shown here. The purple lines (which should be drawn by the students) show the angle of rotation and that the image and preimage are the same distance from the point of rotation (the origin). Point out how a 3-4-5 triangle can be used to find the distance from the origin to \( A \) and \( A' \).

Additional Resources
Differentiated Instruction includes:
- Reading Strategies
- Success for English Learners
- Reteach
- Challenge
9.3 LESSON QUIZ

8.G.1, 8.G.3

Graph triangle $ABC$ with vertices $A(-4, 1)$, $B(-2, 1)$, and $C(-1, -2)$ on a coordinate grid.

1. Graph the image of triangle $ABC$ after a $180^\circ$ clockwise rotation about the origin.

2. Which side of the image is congruent to side $AB$?

3. Which angle in the image is congruent to angle $B$?

4. If a point $M$, located at $(3, -2)$, is rotated clockwise $90^\circ$ about the origin, what are the coordinates of its image, $M$?

5. Angle $G$ in trapezoid $FGHJ$ measures $135^\circ$. If Lee rotates the trapezoid $270^\circ$ counterclockwise about the origin, what will be the measure of angle $G'$ in the image?

Lesson Quiz available online

Answers

1. 

2. $A'B'$$A'B'$$A'B'$

3. angle $B'$

4. $(-2, -3)$

5. $135^\circ$
**9.3 Independent Practice**

7. The figure shows triangle ABC and a rotation of the triangle about the origin.
   a. How would you describe the rotation?
      \( \text{ABC was rotated } 90^\circ \text{ clockwise about the origin.} \)
   b. What are the coordinates of the image?
      \( A' (3, 1), \quad B' (2, 3), \quad C' (-1, 4) \)

8. The graph shows a figure and its image after a transformation.
   a. How would you describe this as a rotation?
      \( \text{The figure was rotated } 180^\circ \text{ about the origin.} \)
   b. Can you describe this as a transformation other than a rotation? Explain.
      \( \text{Yes, you can also describe it as a reflection across the y-axis.} \)

9. What type of rotation will preserve the orientation of the H-shaped figure in the grid?
   \( 180^\circ \text{ rotation about the origin} \)

10. A point with coordinates \((-2, -3)\) is rotated 90° clockwise about the origin. What are the coordinates of its image?
    \( (-3, 2) \)

Complete the table with rotations of 180° or 90°. Include the direction of rotation for rotations of 90°.

<table>
<thead>
<tr>
<th>Shape in quadrant</th>
<th>Image in quadrant</th>
<th>Rotation</th>
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<tr>
<td>I</td>
<td>IV</td>
<td>90° clockwise</td>
</tr>
<tr>
<td>III</td>
<td>I</td>
<td>180°</td>
</tr>
<tr>
<td>IV</td>
<td>III</td>
<td>90° clockwise</td>
</tr>
</tbody>
</table>

**Draw the image of the figure after the given rotation about the origin.**

14. 180°

15. 270° counterclockwise

16. Is there a rotation for which the orientation of the image is always the same as that of the preimage? If so, what?
   \( \text{Yes, a } 360^\circ \text{ rotation} \)

17. **Problem Solving** Lucas is playing a game where he has to rotate a figure for it to fit in an open space. Every time he clicks a button, the figure rotates 90 degrees clockwise. How many times does he need to click the button so that each figure returns to its original orientation?
   - Figure A: 2 times
   - Figure B: 1 time
   - Figure C: 4 times

18. **Make a Conjecture** Triangle ABC is reflected across the y-axis to form the image \( A'B'C' \). Triangle \( A'B'C' \) is then reflected across the x-axis to form the image \( A''B''C'' \). What type of rotation can be used to describe the relationship between triangle \( A''B''C'' \) and triangle ABC?
   \( \text{Triangle } A''B''C'' \text{ is a } 180^\circ \text{ rotation of triangle ABC.} \)

19. **Communicate Mathematical Ideas** Point A is on the y-axis. Describe all possible locations of image \( A' \) for rotations of 90°, 180°, and 270°. Include the origin as a possible location for \( A' \).
    \( \text{Sample answer: If } A \text{ is at the origin, } A' \text{ for any rotation about the origin is at the origin. Otherwise, } A' \text{ is on the } x\text{-axis for } 90^\circ \text{ and } 270^\circ \text{ rotations and on the y-axis for a } 180^\circ \text{ rotation.} \)

**EXTEND THE MATH**

**Activity** The transformed image of point A located at (3, 3) is point \( A' \) located at \((-3, -3)\). Explain how this image could be produced by a translation, a rotation, and by one or more reflections.

By a translation: move 6 units left and 6 units down.

By a rotation: rotate 180° about the origin.

By a reflection or reflections: the point is reflected across the line \( y = -x \), or is reflected across the x-axis and the y-axis sequentially in either order.

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- **Activity available online**
Lesson Support

Content Objective Students will learn how to describe the effect of a translation, rotation, or reflection on coordinates using an algebraic representation.

Language Objective Students will demonstrate how to describe the effect of a translation, rotation, or reflection on coordinates using an algebraic representation.

California Common Core Standards

8.G.3 Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

MP.3 Construct viable arguments and critique the reasoning of others.

Building Background

Visualizing Math Have students work in pairs. Ask them to use triangle ABC and perform the following transformations: translate the triangle 1 unit left and 3 units down; reflect the triangle across the y-axis; rotate the triangle 180°. As they perform the transformations, ask them to look for patterns in how the coordinates of each vertex in the preimage change to create the coordinates of the corresponding vertices in the image. Discuss the patterns students observe.

Learning Progressions

In this lesson, students are introduced to the rules that describe how coordinates change when a figure is transformed by a rigid transformation. Important understandings for students include the following:

- Describe the effect of a translation on the coordinates of the vertices of a geometric figure.
- Describe the effect of a reflection on the coordinates of the vertices of a geometric figure.
- Describe the effect of a rotation on the coordinates of the vertices of a geometric figure.

In earlier lessons, students may have noticed patterns in the way the coordinates of vertices change in rigid transformations. In this lesson they learn the formal rules for the transformations and apply the rules to perform given transformations. They continue to graph the preimage and image to both check and visualize the transformations.

Cluster Connections

This lesson provides an excellent opportunity to connect ideas in the cluster: Understand congruence and similarity using physical models, transparencies, or geometry software.

Triangle QRS has vertices Q(2, 2), R(−2, 3), and S(−2, −2). Ask students to perform the following series of transformations, in order, on triangle QRS.

Rule 1: \((x, y) \rightarrow (x, -y)\)
Rule 2: \((x, y) \rightarrow (-x, y)\)
Rule 3: \((x, y) \rightarrow (x, -y)\)
Rule 4: \((x, y) \rightarrow (-x, y)\)

What is the final result of the transformations? Encourage students to verify their result by graphing the transformations. The final result is the original triangle QRS.
Following Example 2 in the lesson, there is a table that points to the coordinates of the vertices of an image when points are rotated about the origin. English learners may find this especially helpful to tab in their book or to copy into their math journals.

Words that are spelled the same yet have different pronunciations and meanings are called **homographs**. While the word *coordinate* in this lesson refers to one of the values in an ordered pair (x-coordinate or y-coordinate), it also has another meaning and pronunciation as a verb. Discuss these different meanings, and have students add them to their word journals.

**Emerging**  When proficiency in English is limited, having students use their primary language in peer-to-peer discussion encourages higher-level thinking. Have students illustrate rotating a triangle and labeling the vertices.

**Expanding**  Working in small groups is an excellent way for English learners to deepen concept knowledge and practice the academic language and vocabulary. Have students work together to illustrate rotating a triangle and label the new x- and y-coordinates.

**Bridging**  Have students discuss and illustrate rotating a triangle and labeling the new x- and y-coordinates and then take turns explaining how they did it.

**Math Talk**

When you translate a figure to the left, you add to or subtract from the _____.
ESSENTIAL QUESTION
How can you describe the effect of a translation, rotation, or reflection on coordinates using an algebraic representation? Sample answer: For a given transformation, the change in the coordinates can be described algebraically following specific rules for that transformation.

Motivate the Lesson
Ask: How can you find the coordinates of the vertices of an image after a translation, rotation, or reflection without graphing?

Explore
What are the signs of the coordinates of a point in each quadrant of the coordinate plane? How do the coordinates change as you move left and right? up and down?

EXAMPLE 1
Questioning Strategies
• Which coordinate changes and how does it change as you translate a vertex right? Increase Left? Decrease Up? Increase Down? Decrease

• How does the distance between vertices of the image change as compared to the distance between vertices in the preimage? The distances stay the same.

Focus on Modeling
Explain to students that if you are at a 2 on a horizontal number line and move 3 units to the right, you are now at 2 + 3 or 5. A move left of 3 units moves you to 2 − 3 or −1. On a vertical number line, you add when moving up and subtract when moving down.

YOUR TURN
Avoid Common Errors
Students may make changes to the wrong coordinate. Help them understand that a change to the left or right affects the $x$-coordinate, and a change up or down affects the $y$-coordinate.

EXAMPLE 2
Questioning Strategies
• In a reflection over the $x$- or $y$-axis, what are the only ways in which the coordinates will change? One of the coordinates will be multiplied by $-1$.

• Would a translation to the right by 5 units produce the same transformation? No; the labels on the vertices would be different if the rectangle was translated.

Engage with the Whiteboard
Extend the table by two columns, and use the coordinate plane to step through and graph a reflection of RSTU across the $x$-axis.
**LESSON 9.4 Algebraic Representations of Transformations**

**ESSENTIAL QUESTION**
How can you describe the effect of a translation, rotation, or reflection on coordinates using an algebraic representation?

**Algebraic Representations of Transformations**

The rules shown in the table describe how coordinates change when a figure is translated up, down, right, and left on the coordinate plane.

<table>
<thead>
<tr>
<th>Translations</th>
<th>Right a units</th>
<th>Add a to the x-coordinate: (x, y) → (x - a, y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left a units</td>
<td>Subtract a from the x-coordinate: (x, y) → (x - a, y)</td>
<td></td>
</tr>
<tr>
<td>Up b units</td>
<td>Add b to the y-coordinate: (x, y) → (x, y + b)</td>
<td></td>
</tr>
<tr>
<td>Down b units</td>
<td>Subtract b from the y-coordinate: (x, y) → (x, y - b)</td>
<td></td>
</tr>
</tbody>
</table>

**EXAMPLE 1**

Triangle XYZ has vertices X(0, 0), Y(2, 3), and Z(4, -1). Find the vertices of triangle XYZ after a translation of 3 units to the right and 1 unit down. Then graph the triangle and its image.

**STEP 1**
Apply the rule to find the vertices of the image.

<table>
<thead>
<tr>
<th>Vertices of (\triangle XYZ)</th>
<th>Rule: ((x, y)\rightarrow(x - 3, y - 1))</th>
<th>Vertices of (\triangle XYZ')</th>
</tr>
</thead>
<tbody>
<tr>
<td>X(0, 0)</td>
<td>((0 - 3, 0 - 1)) = ((-3, -1))</td>
<td>X'(-3, -1)</td>
</tr>
<tr>
<td>Y(2, 3)</td>
<td>((2 - 3, 3 - 1)) = ((-1, 2))</td>
<td>Y'(-1, 2)</td>
</tr>
<tr>
<td>Z(4, -1)</td>
<td>((4 - 3, -1 - 1)) = ((-1, -2))</td>
<td>Z'(-1, -2)</td>
</tr>
</tbody>
</table>

**Math Talk**
When you translate a figure to the left or right, which coordinate do you change?

**STEP 2**
Graph triangle XYZ and its image.

**EXAMPLE 2**

Rectangle RSTU has vertices R(-4, -1), S(-1, -1), T(-1, -3), and U(-4, -3). Find the vertices of rectangle R'S'T'U' after a reflection across the y-axis. Then graph the rectangle and its image.

<table>
<thead>
<tr>
<th>Vertices of RSTU</th>
<th>Rule: ((-x, y)\rightarrow(-x, y))</th>
<th>Vertices of R'S'T'U'</th>
</tr>
</thead>
<tbody>
<tr>
<td>R(-4, -1)</td>
<td>((-(-4), -1)) = ((-4, -1))</td>
<td>R'(-4, -1)</td>
</tr>
<tr>
<td>S(-1, -1)</td>
<td>((-(-1), -1)) = ((-1, -1))</td>
<td>S'(-1, -1)</td>
</tr>
<tr>
<td>T(-1, -3)</td>
<td>((-(-1), -3)) = ((-1, -3))</td>
<td>T'(-1, -3)</td>
</tr>
<tr>
<td>U(-4, -3)</td>
<td>((-(-4), -3)) = ((-4, -3))</td>
<td>U'(-4, -3)</td>
</tr>
</tbody>
</table>

**YOUR TURN**

1. A rectangle has vertices at (0, -2), (0, 3), (3, -2), and (3, 3). What are the coordinates of the vertices of the image after the translation \((x, y)\rightarrow(x - 6, y - 3)\)? Describe the translation. \((-6, -5), (-6, 0), (-3, -5), \text{and} (-3, 0)\) the rectangle is translated 6 units to the left and 3 units down.

**PROFESSIONAL DEVELOPMENT**

**Integrate Mathematical Practices MP.3**
This lesson provides an opportunity to address this Mathematical Practices standard. It calls for students to use logic to analyze situations. Students use the rules for translations, reflections, and rotations to find the vertices of the image using an algebraic representation instead of graphs. Also, students use an algebraic rule to create a graph of an image, then use the graph to describe the transformation.

**Math Background**
Having a good understanding of transformations in both their graphical and algebraic representations will be beneficial for students in more advanced levels of algebra. For example, students will be using a parabola that has a vertex at the origin, and then translating it while maintaining its shape. Students will use the equation for the original parabola and the translation rule to write the equation of the translated parabola.
ADDITIONAL EXAMPLE 2
Triangle PQR has vertices P(3, 3), Q(5, −1), and R(1, −3). Find the vertices of triangle P′Q′R′ after a reflection across the y-axis. Then graph the triangle and its image.

**YOUR TURN**

Avoid Common Errors
Make sure that students understand they do not just make the y-value a negative number. The y-value of the image must have a sign opposite that of the preimage’s y-value.

EXAMPLE 3

**Questioning Strategies**

Mathematical Practices

- In a rotation of 180°, what is the relationship between the coordinates of the preimage and the coordinates of the image? The numbers are the same, but the signs of the coordinates of the image are the opposite of the signs of the coordinates of the preimage.

- If the point (−2, 5) is rotated 90° clockwise, what are the new coordinates? (5, 2) What if (−2, 5) is rotated 180°? (2, −5)

**Focus on Modeling**

Mathematical Practices

In Step 2, be sure students understand why D and D′ are plotted at the same point.

**Integrating Language Arts**

Encourage a broad class discussion on the Reflect. English learners will benefit from hearing and participating in classroom discussions.

**YOUR TURN**

Avoid Common Errors
It does not matter if students multiply the y-values by −1 before or after the coordinates are switched in Exercise 4, but multiplying by −1 first may prevent errors.

**Elaborate**

**Talk About It**

**Summarize the Lesson**

Ask: Is it possible to determine if a translation, reflection, or rotation occurred by examining the coordinates of the image and preimage? The rules for translations, reflections, and rotations affect coordinates of the ordered pair in distinct ways, so it is often possible to determine which transformation occurred.

**GUIDED PRACTICE**

**Engage with the Whiteboard**

For Exercise 2, graph a point such as (2, 2) and reflect it across the x-axis on the grid for Exercise 1.

**Avoid Common Errors**

**Exercise 3** Remind students that they must state the number of degrees, point of rotation, and the direction of the rotation.
Algebraic Representations of Transformations

When points are rotated about the origin, the coordinates of the image can be found using the rules shown in the table.

<table>
<thead>
<tr>
<th>Rotations</th>
<th>x-coordinate change</th>
<th>y-coordinate change</th>
</tr>
</thead>
<tbody>
<tr>
<td>90° clockwise</td>
<td>x → -y</td>
<td>y → x</td>
</tr>
<tr>
<td>90° counterclockwise</td>
<td>x → y</td>
<td>y → -x</td>
</tr>
<tr>
<td>180°</td>
<td>x → -x</td>
<td>y → -y</td>
</tr>
</tbody>
</table>

**EXAMPLE 3**

Quadrilateral ABCD has vertices at A(−4, 2), B(−3, 4), C(2, 3), and D(0, 0). Find the vertices of quadrilateral A'B'C'D' after a 90° clockwise rotation.

**STEP 1**
Apply the rule to find the vertices of the image.

- **Vertices of ABCD**
  - A(−4, 2)
  - B(−3, 4)
  - C(2, 3)
  - D(0, 0)

- **Rule:** (x, y) → (y, −x)

- **Vertices of A'B'C'D'**
  - A'(2, −4)
  - B'(4, −3)
  - C'(3, −2)
  - D'(0, 0)

**STEP 2**
Graph the quadrilateral and its image.

---

**YOUR TURN**

2. Triangle ABC has vertices A(−2, 6), B(0, 5), and C(3, −1). Find the vertices of triangle A'B'C' after a reflection across the x-axis.

   \[ A'(-2, -6), B'(0, -5), \text{and } C'(3, 1) \]

---

**Reflect**

3. Communicate Mathematical Ideas
   How would you find the vertices of an image if a figure were rotated 270° clockwise? Explain.

   A 270° clockwise rotation is the same as a 90° counterclockwise rotation. Change the sign of the y-coordinate and switch the coordinates.

4. A triangle has vertices at A(−2, −4), B(1, 5), and C(2, 2). What are the coordinates of the vertices of the image after the triangle is rotated 90° clockwise?

   \[ J(4, -2), K(-5, 1), \text{and } L(-2, 2) \]

---

**DIFFERENTIATE INSTRUCTION**

**Communicating Math**
Have students work in pairs. One student provides a transformation. The other describes how the ordered pairs change using an algebraic representation.

- **Student 1:** Translate right 3 units and up 2 units.
- **Student 2:** Add 3 to the x-value, and add 2 to the y-value: \((x, y) \rightarrow (x + 3, y + 2)\).

**Visual Cues**
On three separate index cards, have students write the rules for how to change the coordinates for a figure when it is translated, rotated, and reflected. Have students use colored pencils or markers to emphasize the changes made to the x- and y-values.

**Additional Resources**
Differentiated Instruction includes:
- Reading Strategies
- Success for English Learners
- Reteach
- Challenge
**Evaluate**

**GUIDED AND INDEPENDENT PRACTICE**

<table>
<thead>
<tr>
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<th>Practice</th>
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</thead>
<tbody>
<tr>
<td><strong>Example 1</strong></td>
<td><strong>Exercises 1, 5, 7, 9–11</strong></td>
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<tr>
<td>Algebraic Representations of Translations</td>
<td></td>
</tr>
<tr>
<td><strong>Example 2</strong></td>
<td><strong>Exercises 2, 8</strong></td>
</tr>
<tr>
<td>Algebraic Representations of Reflections</td>
<td></td>
</tr>
<tr>
<td><strong>Example 3</strong></td>
<td><strong>Exercises 3, 6, 12</strong></td>
</tr>
<tr>
<td>Algebraic Representations of Rotations</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Exercise</strong></th>
<th><strong>Depth of Knowledge (D.O.K.)</strong></th>
<th><strong>Mathematical Practices</strong></th>
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</thead>
<tbody>
<tr>
<td>5–9</td>
<td>2 Skills/Concepts</td>
<td>MP.6 Precision</td>
</tr>
<tr>
<td>10</td>
<td>2 Skills/Concepts</td>
<td>MP.7 Using Structure</td>
</tr>
<tr>
<td>11–12</td>
<td>2 Skills/Concepts</td>
<td>MP.6 Precision</td>
</tr>
<tr>
<td>13</td>
<td>3 Strategic Thinking</td>
<td>H.O.T. MP.2 Reasoning</td>
</tr>
<tr>
<td>14</td>
<td>3 Strategic Thinking</td>
<td>H.O.T. MP.3 Logic</td>
</tr>
<tr>
<td>15</td>
<td>3 Strategic Thinking</td>
<td>H.O.T. MP.2 Reasoning</td>
</tr>
</tbody>
</table>

**Additional Resources**

Differentiated Instruction includes:
- Leveled Practice worksheets

**Exercise 13** combines concepts from the California Common Core cluster "Understand congruence and similarity using physical models, transparencies, or geometry software.”

---

**9.4 LESSON QUIZ**

**CA CC 8.G.3**

Triangle $ABC$ has vertices $A(-4, 1)$, $B(-2, 1)$, and $C(-1, -2)$.

1. Find the coordinates of the vertices of triangle $A'B'C'$ after a 90° clockwise rotation about the origin.

2. Find the coordinates of the vertices of triangle $A'B'C'$ after triangle $ABC$ is reflected across the $y$-axis.

3. Find the coordinates of the vertices of triangle $A'B'C'$ after triangle $ABC$ is translated using the rule $(x, y) \rightarrow (x + 5, y - 3)$. Then describe the translation.

4. Point $M$ has coordinates $(3, -2)$. The coordinates of point $M'$ after a single transformation are $(-3, 2)$. Name a transformation that could have done this.

**Answers**

1. $A'(1, 4), B'(1, 2),$ and $C'(-2, 1)$

2. $A'(4, 1), B'(2, 1),$ and $C'(1, -2)$

3. $A'(1, -2), B'(3, -2),$ and $C'(4, -5)$; The triangle is translated 5 units to the right and 3 units down.

4. Sample answers: translation 6 units left and 4 units up; rotation of 180°; reflection across the line $y = x$
**Activity**
Transform the square shown into five smaller, but equal squares, with only four cuts. The sum of the areas of the five smaller squares must total the area of the larger square. Pieces of the larger square, produced when the four cuts are made, can be rotated, reflected, or translated and combined to form the five smaller squares.

**EXTEND THE MATH**

**Activity available online**
my.hrw.com

**Focus on Higher Order Thinking**

11. Write an algebraic rule to describe each transformation. Then describe the transformation.

5. \((x, y) \rightarrow (x - 2, y - 3); \) translation of 2 units to the left and 3 units down

6. \((x, y) \rightarrow (-x, y); \) rotation of \(180^\circ\)

7. Triangle XYZ has vertices X(6, −2.3), Y(7.5, 5), and Z(8.4, 4). When translated, X′ has coordinates (2.8, −1.3). Write a rule to describe this transformation. Then find the coordinates of Y′ and Z′.

\(x, y \rightarrow (x - 3.2, y + 1); \) Y′(4.3, 6), Z′(4.8, 5)

8. Point L has coordinates (3, −5). The coordinates of point L′ after a reflection are (−3, −5). Without graphing, tell which axis point L was reflected across. Explain your answer.

y-axis; when you reflect a point across the y-axis, the sign of the x-coordinate changes and the sign of the y-coordinate remains the same.

9. Use the rule \((x, y) \rightarrow (x - 2, y - 4); \) to graph the image of the rectangle. Then describe the transformation.

The rectangle is translated 2 units to the left and 4 units down.

10. Parallelogram ABCD has vertices A(−2, −2.5), B(−4, −5.5), C(−3, −2), and D(−1, −2). Find the vertices of parallelogram A′B′C′D′ after a translation of \(2 \frac{1}{2}\) units down. A′(−2, −8), B′(−4, −10), C′(−3, −4.5), and D′(−1, −6.5)

11. Alexandra drew the logo shown on half-inch graph paper. Write a rule that describes the translation Alexandra used to create the shadow on the letter A.

\((x, y) \rightarrow (x + 0.5, y - 0.25)\)

12. Kite KLMN has vertices at K(1, 3), L(2, 4), M(3, 3), and N(2, 0). After the kite is rotated, K′ has coordinates (−3, 1). Describe the rotation, and include a rule in your description. Then find the coordinates of L′, M′, and N′. 90° counterclockwise; \((x, y) \rightarrow (-y, x); L′(-4, 2), M′(-3, 3), N′(0, 2)\)

13. Make a Conjecture

Graph the triangle with vertices (−3, 4), (3, 4), and (−5, −5). Use the transformation \((y, x)\) to graph its image.

a. Which vertex of the image has the same coordinates as a vertex of the original figure? Explain why this is true.

(−5, −5); x and y are equal, so switching x and y has no effect on the coordinates.

b. What is the equation of a line through the origin and this point?

\(y = x\)

c. Describe the transformation of the triangle.

The triangle is reflected across the line \(y = x\).

14. Critical Thinking

Mitchell says the point (0, 0) does not change when reflected across the x- or y-axis or when rotated about the origin. Do you agree with Mitchell? Explain why or why not.

Yes; reflecting across the x- or y-axis changes the sign of the y- or x-coordinate; 0 cannot change signs. Rotating about the origin doesn’t change the origin, (0, 0).

15. Analyze Relationships

Triangle ABC with vertices A(−2, −2), B(−3, 1), and C(1, 1) is translated by \((x, y) \rightarrow (x - 1, y + 3)\). Then the image, triangle A′B′C′, is translated by \((x, y) \rightarrow (x + 4, y - 1)\), resulting in A′B′C″.

a. Find the coordinates for the vertices of triangle A′B′C″.

A′(1, 0), B′(0, 3), and C′(4, 3)

b. Write a rule for one translation that maps triangle ABC to triangle A′B′C″.

\((x, y) \rightarrow (x + 3, y + 2)\)
Lesson Support

**Content Objective**  Students will learn how transformations can be used to verify that two figures have the same shape and size.

**Language Objective**  Students will show how transformations can be used to verify that two figures have the same shape and size.

---

**California Common Core Standards**

- **8.G.2** Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

- **MP.6** Attend to precision.

---

**Building Background**

**Visualizing Math**  Have students create a pattern by using a series of rigid transformations. Ask them to draw a simple geometric figure, such as a triangle, quadrilateral, or pentagon. Then ask them to translate, reflect, or rotate the figure several times to create a pattern. It may be helpful to have students use graph paper. Encourage them to combine transformations. For example, they can create a slide image by reflecting and translating the figure successively. Discuss whether the figures in the pattern are congruent and why or why not.

---

**Learning Progressions**

In this lesson, students are introduced to a formal definition of congruence in the coordinate plane. Important understandings for students include the following:

- Combine transformations to create a congruent figure.
- Identify a sequence of transformations that create a congruent figure.

Students conclude the work with rigid transformations that they have been studying throughout this module with this lesson. They understand that two figures in the plane are congruent if one can be transformed into the other using a series of rigid transformations. They also apply their experience with the transformations to describe a sequence that will transform one figure into the other in the coordinate plane.

---

**Cluster Connections**

This lesson provides an excellent opportunity to connect ideas in the cluster: Understand congruence and similarity using physical models, transparencies, or geometry software. Have students graph triangle ABC with vertices A(−1, 1), B(−3, 2), and C(−3, 1). Then ask them to perform the following series of transformations, in order, on triangle ABC.

1. Rotate the triangle 90° clockwise.
2. Reflect the rotated triangle over the x-axis.
3. Translate the reflected triangle 3 units left and 1 unit up.

What are the coordinates of the vertices of the final image?

(−2, 0), (−1, −2), (−2, −3)
This lesson opens with an Explore Activity in which students are provided instructions for combining transformations. Understanding detailed instructions like these requires a high level of English proficiency. To assure that English learners catch the details described, you may want to form groups of students of mixed English proficiency levels to solve problems. Provide them with sentence frames to support their responses. Encourage students to use visuals and check each step as information is shared.

When English learners read the instructions on tests and in textbooks, they encounter the imperative form or command form. For example, if the instructions tell the student *complete*, it means they are to finish out the exercise. Among the verbs in the imperative form in this lesson are *apply, label, compare, identify, graph,* and *describe.* Have English learners add these to their word journals for future reference.

When proficiency in English is limited, having students use their primary language in peer-to-peer discussion encourages higher-level thinking. Have students check the glossary, review the definition of *congruent figures,* and then illustrate and label an example on graph paper.

Have students at this level of English proficiency work in pairs to review the definition of *congruent figures* and then illustrate examples of a congruent figures with different transformations on graph paper.

Have students at this level of English proficiency work in pairs to review and rephrase the definition of *congruent figures* and then illustrate examples of congruent figures with different transformations on graph paper.

Write out and model for students a sentence frame to begin their answer.

The sequence of transformations must include a rotation when _______.

---

**California ELD Standards**

<table>
<thead>
<tr>
<th>Level</th>
<th>Standard</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Emerging</strong></td>
<td>2.II.3.</td>
<td>Using verbs and verb phrases – Use a variety of verbs in different tenses and aspects appropriate for the text type and discipline on familiar topics.</td>
</tr>
<tr>
<td><strong>Expanding</strong></td>
<td>2.II.3.</td>
<td>Using verbs and verb phrases – Use a variety of verbs in different tenses and aspects appropriate for the task, text type, and discipline on an increasing variety of topics.</td>
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<td><strong>Bridging</strong></td>
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ESSENTIAL QUESTION

How can transformations be used to verify that two figures have the same shape and size?

Sample answer: If two figures have the same shape and size, then there exists a sequence of translations, reflections, and/or rotations that transforms one into the other.

Motivate the Lesson

Ask: What effect will a combination of rotations, translations, and/or reflections have on a triangle’s size and shape? Take a guess. Begin the Explore Activity to find out.

Explore

EXPLORE ACTIVITY

Engage with the Whiteboard

Color coding the corresponding sides or labeling the vertices of the triangle and its images will help students better visualize the movement of the triangle.

Example

EXAMPLE 1

Questioning Strategies

- How do you know when a rotation is part of the transformation? The image will be turned when compared to the original figure.
- In Parts B and C, why do you think the rotation is performed before the translation? Sample answer: It can be difficult to see exactly where the figure will end up after a rotation. By rotating first, you can simply translate the figure into the correct place, avoiding this issue.

Engage with the Whiteboard

For each part, have a volunteer graph the intermediate step in producing the final image. For example, in Part A, the volunteer would graph the reflection of figure $A$ over the $y$-axis before it is translated 1 unit left.

Focus on Communication

In Part C, encourage students to suggest other approaches that might map figure $D$ on to figure $E$. For example, figure $D$ could have been reflected over the $x$-axis, rotated $90^\circ$ clockwise about the origin, and translated 1 unit down. The algebraic sequence of transformations is $(x, y) \rightarrow (-x, y), (x, y) \rightarrow (y, -x), (x, y) \rightarrow (x, y - 1)$.
**Lesson 9.5 Congruent Figures**

**Essential Question**
How can transformations be used to verify that two figures have the same shape and size?

**Explore Activity**

**Combining Transformations**
Apply the indicated series of transformations to the triangle. Each transformation is applied to the image of the previous transformation, not the original figure. Label each image with the letter of the transformation applied.

- **A** Reflection across the x-axis
- **B** $(x,y) \rightarrow (x - 3, y)$
- **C** Reflection across the y-axis
- **D** $(x,y) \rightarrow (x, y + 4)$
- **E** Rotation 90° clockwise around the origin
- **F** Compare the size and shape of the final image to that of the original figure.

**Reflect**
1. Which transformation(s) change the orientation of figures? Which do not?
   Reflections and rotations; translations

2. **Make a Conjecture**
   After a series of transformations, two figures have the same size and shape, but different orientations. What does this indicate about the transformations?
   None of the transformations had properties that effected the congruency of the figures, but at least one of the transformations was a reflection or a rotation.

**Example 1**

**Identify a sequence of transformations that will transform figure A into figure B.**

In both cases, the image is turned sideways from the original figure. Only a rotation will do this.

**Math Talk**

How do you know that the sequence of transformations in Parts B and C must include a rotation?

Any sequence of transformations that changes figure B into figure C will need to include a rotation. A 90° counterclockwise rotation around the origin would properly orient figure B, but not locate it in the same position as figure C. The rotated figure would be 2 units below and 1 unit to the left of where figure C is. You would need to translate the rotated figure up 2 units and right 1 unit.

A sequence of transformations is a 90° counterclockwise rotation about the origin, $(x,y) \rightarrow (-y, x)$, followed by $(x,y) \rightarrow (x + 1, y + 2)$.

**Professional Development**

**Integrate Mathematical Practices MP.6**
This lesson provides an opportunity to address this Mathematical Practice standard. It calls for students to attend to precision. Students pay close attention to the coordinates of the vertices of a figure in order to apply a given sequence of transformations and graph the resulting image. Each transformation must be carefully and precisely applied to obtain the desired outcome. Students also must pay close attention to the coordinates of the vertices of a figure and its images when determining the sequence of transformations that result in a figure being transformed into a particular image.

**Math Background**
If there exists a sequence of translations, reflections, and/or rotations that will transform one figure into the other, the two figures are congruent. Note that dilations are not included in this list of transformations. The image after a dilation is either an enlargement or reduction of the original figure, making the two figures similar but not congruent. While dilations preserve the shape of a figure, they do not preserve the size. A dilation is often called a similarity transformation.

**Congruent Figures**
Recall that segments and their images have the same length and angles and their images have the same measure under a translation, reflection, or rotation. Two figures are said to be congruent if one can be obtained from the other by a sequence of translations, reflections, and rotations. Congruent figures have the same size and shape.

When you are told that two figures are congruent, there must be a sequence of transformations, reflections, and/or rotations that transforms one into the other.
YOUR TURN

Avoid Common Errors

If students cannot visualize the sequence of transformations, suggest they use a cut-out paper triangle the same size and shape as figure A that they can rotate and translate on the coordinate grid.

Talk About It

Check for Understanding

Ask: How do you know if a two-dimensional figure is congruent to another? Two figures are congruent if the second can be obtained from the first by a sequence of rotations, reflections, and translations. The two figures will have the same size and shape.

Elaborate

Talk About It

Summarize the Lesson

Ask: When two figures have different orientations, what clues help you decide which transformations were performed in the sequence of transformations? In order to get a turned image, a rotation must have occurred. In order to get a mirror image, a reflection must have occurred.

GUIDED PRACTICE

Engage with the Whiteboard

For Exercise 1, have five volunteers take turns drawing the indicated series of transformations on the coordinate grid provided.

Avoid Common Errors

Exercise 2  Students might think a rotation was used to transform figure A into figure B. Suggest students use a paper cut-out of triangle A and actually rotate it about point (0, 2) to see that the orientation of figure B is not right for the transformation to have been a rotation.

Exercise 4  Before trying to write the algebraic sequence of transformations used, suggest that students label figures A, B, and C with the ordered pairs for each of the vertices. Then, analyze the pairs to understand the changes to the x and y values.
Differentiate Instruction

Multiple Representations
Provide students with 6 congruent equilateral triangles. Have students arrange the triangles to form a hexagon. Students should then draw a simple but colorful design on one of the triangles and then reflect that design around the hexagon 5 times onto the other triangles. Explain that their final design will be a kaleidoscope image, as shown here.

Additional Resources
Differentiated Instruction includes:
- Reading Strategies
- Success for English Learners
- Reteach
- Challenge
9.5 LESSON QUIZ  8.G.2

1. On a coordinate grid, graph a triangle with its vertices at (−2, 1), (−4, 1), and (−1, 4). Then apply the indicated series of transformations to the triangle. Each transformation is applied to the image of the previous transformation. Label each image with the letter of the transformation applied.
   A Rotation 90º clockwise around the origin
   B $(x, y) \rightarrow (x, y - 3)$
   C $(x, y) \rightarrow (x - 3, y - 2)$

2. Identify a sequence of transformations that will transform figure A into figure B.

Exercise Depth of Knowledge (D.O.K.) Mathematical Practices

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<tr>
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Additional Resources
Differentiated Instruction includes:
- Leveled Practice worksheets

Answers
1.

2. Reflection across the $y$-axis, rotation 90º clockwise about the origin; $(x, y) \rightarrow (-x, y), (x, y) \rightarrow (y, -x)$
9.5 Independent Practice

For each given figure A, graph figures B and C using the given sequence of transformations. State whether figures A and C have the same or different orientation.

7. Figure A: a translation of 1 unit to the right and 3 units up
   Figure B: a translation of 1 unit to the right and 3 units up
   Figure C: a 90° clockwise rotation around the origin
   Different orientation

8. Figure A: a reflection across the y-axis
   Figure B: a reflection across the y-axis
   Figure C: a 180° rotation around the origin
   Different orientation

9. Figure A: a reflection across the y-axis
   Figure B: a translation 2 units up
   Figure C: a translation 2 units down
   Different orientation

10. Figure A: a translation 4 units left
    Figure B: a translation 2 units right and 1 unit up
    Figure C: a rotation of 180° around the origin
    Different orientation

EXTEND THE MATH

Activity A translation followed by a reflection about a line that is parallel to the line of translation is called a glide reflection. The heart shown below has been glided and reflected twice.

Draw a horizontal line on a piece of paper, choose a shape, and make a glide reflection pattern.
9.1–9.3 Properties of Translations, Reflections, and Rotations

1. Graph the image of triangle $ABC$ after a translation of 6 units to the right and 4 units down. Label the vertices of the image $A', B'$, and $C'$.

2. On the same coordinate grid, graph the image of triangle $ABC$ after a reflection across the $x$-axis. Label the vertices of the image $A'', B''$, and $C''$.

3. Graph the image of $HIJK$ after it is rotated 180° about the origin. Label the vertices of the image $H'I'J'K'$.

9.4 Algebraic Representations of Transformations

4. A triangle has vertices at $(2, 3)$, $(-2, 2)$, and $(-3, 5)$. What are the coordinates of the vertices of the image after the translation $(x, y) \rightarrow (x + 4, y - 3)$?

9.5 Congruent Figures

5. Vocabulary
Translations, reflections, and rotations produce a figure that is congruent to the original figure.

6. Use the coordinate grid for Exercise 3. Reflect $H'I'J'K'$ over the $y$-axis, then rotate it 180° about the origin. Label the new figure $H''I''J''K''$.

7. What properties allow transformations to be used as problem solving tools?

Sample answer: Transformational properties allow systematic movement of congruent figures while maintaining or adjusting their orientation.
1. Consider each rational number. Is the number greater than $-2 \frac{1}{3}$ but less than $-\frac{4}{5}$?
   Select Yes or No for A–C.
   A. $-0.4$ Yes No
   B. $-\frac{9}{7}$ Yes No
   C. $-0.9$ Yes No

2. Triangle $DEF$ is rotated $90^\circ$ clockwise about the origin. Choose True or False for each statement.
   A. $E'F' = 3$ True False
   B. $m\angle F' = 90^\circ$ True False
   C. $D'F'$ is vertical. True False

3. A graphic artist is working on a logo for a car company. The artist draws parallelogram $LMNP$ with vertices $L(1, 0)$, $M(2, 2)$, $N(5, 3)$, and $P(4, 1)$. The artist then reflects the parallelogram across the $x$-axis. Find the coordinates of the vertices of parallelogram $L'M'N'P'$, and describe the algebraic rule you used to find the coordinates.

4. The map shows two of a farmer’s fields. Is field $A$ congruent to field $B$? Use a sequence of transformations to explain your reasoning.

Yes; sample answer: You can transform field $A$ into field $B$ by rotating field $A$ $90^\circ$ clockwise about the origin and then translating it 7 units down. Because you can transform field $A$ into field $B$ using only a rotation and a translation, the fields are congruent.

**California Common Core Standards**

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* Item integrates mixed review concepts from previous modules or a previous course.