Combinations and Probability

Essential Question: What is the difference between a permutation and a combination?

You can choose a number of objects in such a way that the order matters, in which case you choose a permutation, or you can choose in such a way that order does not matter, in which case you choose a combination.

**PREVIEW: LESSON PERFORMANCE TASK**

View the Engage section online. Discuss the photograph. Ask students to guess the location where the photo was taken and describe what is happening there. Then preview the Lesson Performance Task.

**Finding the Number of Combinations**

A combination is a selection of objects from a group in which order is unimportant. For example, if 3 letters are chosen from the group of letters A, B, C, and D, there are 4 different combinations:

<table>
<thead>
<tr>
<th>Combination</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABC</td>
</tr>
<tr>
<td>ABD</td>
</tr>
<tr>
<td>ACD</td>
</tr>
<tr>
<td>BCD</td>
</tr>
</tbody>
</table>

A restaurant has 8 different appetizers on the menu, as shown in the table. They also offer an appetizer sampler, which contains any 3 of the appetizers served on a single plate. How many different appetizer samplers can be created? The order in which the appetizers are selected does not matter.

**Appetizers**

<table>
<thead>
<tr>
<th>Appetizers 1</th>
<th>Appetizers 2</th>
<th>Appetizers 3</th>
<th>Appetizers 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nachos</td>
<td>Chicken Wings</td>
<td>Soft Pretzels</td>
<td>Guacamole Dip</td>
</tr>
<tr>
<td>Chicken Quesadilla</td>
<td>Vegetarian Egg Rolls</td>
<td>Beef Chili</td>
<td></td>
</tr>
<tr>
<td>Potato Skins</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beef Chili</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. Find the number appetizer samplers that are possible if the order of selection does matter. This is the number of permutations of 8 objects taken 3 at a time.

\[ P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5!}{5!} = 336 \]

2. Find the number of different ways to select a particular group of appetizers. This is the number of permutations of 3 objects.

\[ P_3 = \frac{3!}{(3-3)!} = \frac{3!}{0!} = 6 \]
To find the number of possible appetizer samplers if the order of selection does not matter, divide the answer to part A by the answer to part B.

So the number of appetizer samplers that can be created is \( \frac{336}{6} = 56 \).

**Reflect**

1. Explain why the answer to Part A was divided by the answer to Part B.

   Since the order of selection does not matter, the answer to Part A contained duplications of each possible sampler. Dividing by the answer to Part B removed the duplicates of each sampler.

2. On Mondays and Tuesdays, the restaurant offers an appetizer sampler that contains any 4 of the appetizers listed. How many different appetizer samplers can be created? The number of appetizer samplers that can be created is \( \binom{8}{3} = \frac{8!}{3!(8-3)!} = 56 \).

3. In general, are there more ways or fewer ways to select objects when the order does not matter? Why?

   There are fewer ways to select objects when the order does not matter. This is because multiple selections are counted as the same combination.

**Explain 1** Finding a Probability Using Combinations

The results of the Explore can be generalized to give a formula for combinations. In the Explore, the number of combinations of the 8 objects taken 3 at a time is

\[ \binom{8}{3} = \frac{8!}{3!(8-3)!} = \frac{8!}{3!5!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56 \]

This can be generalized as follows.

The number of combinations of \( n \) objects taken \( r \) at a time is given by

\[ \binom{n}{r} = \frac{n!}{r!(n-r)!} \]

**Example 1** Find each probability.

There are 4 boys and 8 girls on the debate team. The coach randomly chooses 3 of the students to participate in a competition. What is the probability that the coach chooses all girls?

The sample space \( S \) consists of combinations of 3 students taken from the group of 12 students.

\[ n(S) = \binom{12}{3} = \frac{12!}{3!(12-3)!} = 220 \]

Event \( A \) consists of combinations of 3 girls taken from the set of 8 girls.

\[ n(A) = \binom{8}{3} = \frac{8!}{3!(8-3)!} = 56 \]

The probability that the coach chooses all girls is

\[ P(A) = \frac{n(A)}{n(S)} = \frac{56}{220} = \frac{14}{55} \]

EXPLAIN 1
Finding a Probability Using Combinations

**INTEGRATE TECHNOLOGY**

Graphing calculators have a built-in function that calculates combinations. The \( \binom{n}{r} \) function is available from the MATH PRB menu. Have students enter data in the same way they enter \( n \) and \( r \) when applying the permutation function. For example, to find \( \binom{8}{3} \), first enter 6. Then press MATH and use the arrow keys to choose the PRB menu. Select \( 3:nCr \) and press ENTER. Now enter 4 and press ENTER to see that \( \binom{8}{3} = 56 \).

**Math Background**

In this lesson students are introduced to combinations. A combination of \( n \) objects taken \( r \) at a time is given by the rule \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \). Pascal’s triangle is a number triangle with rows arranged according to the combination formula, starting with \( n = 0 \) and continuing indefinitely. Each successive row can be found by adding elements from the row above. It has many fascinating properties, and it can be used as a shortcut in algebra when factoring polynomials.
There are 52 cards in a standard deck, 13 in each of 4 suits: clubs, diamonds, hearts, and spades. Five cards are randomly drawn from the deck. What is the probability that all five cards are diamonds?

The sample space $S$ consists of combinations of 52 cards drawn from 52 cards.

$\text{n}(S) = \binom{52}{5} = \frac{52!}{5!(52-5)!} = 2,598,960$

Event $A$ consists of combinations of 5 cards drawn from the 13 diamonds.

$\text{n}(A) = \binom{13}{5} = \frac{13!}{5!(13-5)!} = 1287$

The probability of randomly selecting 5 cards that are diamonds is

$P(A) = \frac{\text{n}(A)}{\text{n}(S)} = \frac{1287}{2,598,960} = \frac{33}{66,640}$

Your Turn

4. A coin is tossed 4 times. What is the probability of getting exactly 3 heads?

The number of outcomes in the sample space $S$ is found by using the Fundamental Counting Principle since each flip can result in heads or tails.

$\text{n}(S) = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$

Event $A$ consists of combinations of 3 heads taken from the set of 4 coin flips, so

$\text{n}(A) = \binom{4}{3} = \frac{4!}{3!(4-3)!} = 4$

The probability of getting exactly 3 heads is

$P(A) = \frac{\text{n}(A)}{\text{n}(S)} = \frac{4}{16} = \frac{1}{4}$

5. A standard deck of cards is divided in half, with the red cards (diamonds and hearts) separated from the black cards (spades and clubs). Four cards are randomly drawn from the red half. Is the probability they are all diamonds?

The sample space $S$ consists of combinations of 4 cards drawn from the 26 red cards, so

$\text{n}(S) = \binom{26}{4} = \frac{26!}{4!(26-4)!} = 14,950.$

Event $A$ consists of combinations of 4 cards drawn from the 13 diamonds, so

$\text{n}(A) = \binom{13}{4} = \frac{13!}{4!(13-4)!} = 715.$

The probability of getting all diamonds is

$P(A) = \frac{\text{n}(A)}{\text{n}(S)} = \frac{715}{14,950} = \frac{11}{230}$

COLLABORATIVE LEARNING

Small Group Activity

Have students create two situations in which 2 out of 16 objects are selected, with order important in one of the situations and not important in the other. Ask students to use the situations to compare and contrast permutations and combinations, explaining why the number of selections is greater in one than in the other, and describing the relationship between the two. Tell students it may be necessary to create several situations in which order is and is not important to be able to describe the relationship between permutations and combinations.
Explain 2 Finding a Probability Using Combinations and Addition

Sometimes, counting problems involve the phrases “at least” or “at most.” For these problems, combinations must be added.

For example, suppose a coin is flipped 3 times. The coin could show heads 0, 1, 2, or 3 times. To find the number of combinations with at least 2 heads, add the number of combinations with 2 heads and the number of combinations with 3 heads \( \binom{3}{2} + \binom{3}{3} \).

Example 2 Find each probability.

A coin is flipped 5 times. What is the probability that the result is heads at least 4 of the 5 times?

The number of outcomes in the sample space \( S \) can be found by using the Fundamental Counting Principle since each flip can result in heads or tails.

\[ n(S) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32 \]

Let \( A \) be the event that the coin shows heads at least 4 times. This is the sum of 2 events, the coin showing heads 4 times and the coin showing heads 5 times. Find the sum of the combinations with 4 heads from 5 coins and with 5 heads from 5 coins.

\[ n(A) = \binom{5}{4} + \binom{5}{5} = \frac{5!}{4!1!} + \frac{5!}{5!0!} = 5 + 1 = 6 \]

The probability that the coin shows at least 4 heads is

\[ P(A) = \frac{n(A)}{n(S)} = \frac{6}{32} = \frac{3}{16} \]

Three number cubes are rolled and the result is recorded. What is the probability that at least 2 of the number cubes show 6?

The number of outcomes in the sample space \( S \) can be found by using the Fundamental Counting Principle since each roll can result in 1, 2, 3, 4, 5, or 6.

\[ n(S) = 6^3 = 216 \]

Let \( A \) be the event that at least 2 number cubes show 6. This is the sum of 2 events, 2 number cubes showing 6 or 3 number cubes showing 6. The event of getting 6 on 2 number cubes occurs \( \binom{2}{2} \) times since there are \( \frac{2!}{2!0!} \) possibilities for the other number cube.

\[ n(A) = \binom{3}{2} \cdot \binom{3}{2} + \binom{3}{3} = \frac{3!}{2!1!} \cdot \frac{3!}{2!0!} = 3 + 1 = 4 \]

The probability of getting a 6 at least twice in 3 rolls is

\[ P(A) = \frac{n(A)}{n(S)} = \frac{216}{216} = \frac{2}{27} \]

DIFFERENTIATE INSTRUCTION

Graphic Organizers

Have the class work together to create a graphic organizer that summarizes and demonstrates the rules for counting: the Fundamental Counting Principle, the permutation rule, and the combination rule. For each rule, have students include the formula and an example. Suggest that students highlight words in the examples that indicate why and how the rule is applied.

AVOID COMMON ERRORS

Students may have difficulty using combinations to find probability. Have students break down the probability problem into parts. First find the size of the sample space. Then find the number of outcomes associated with the event. Finally, write the ratio.

EXPLAIN 2

Finding a Probability Using Combinations and Addition

INTEGRATE MATHEMATICAL PRACTICES

Focus on Communication

MP.3 Students may have difficulty recognizing how to use addition with combinations, and understanding why addition is used. Review how to find the probability of simple events that involve addition when rolling a number cube, such as rolling at least a 3 or at most a 4.

QUESTIONING STRATEGIES

When you find a probability using combinations, will both parts of the probability ratio necessarily be combinations? Explain. No, the method used for counting each part of the ratio depends on the problem.
ELABORATE

AVOID COMMON ERRORS

Students sometimes compute a combination for \( n(S) \) and then choose the numerator of the probability ratio carelessly. Emphasize the importance of accurately identifying both parts of the probability ratio.

SUMMARIZE THE LESSON

What are combinations and how can you use them to calculate probabilities? A combination is a grouping of objects in which order does not matter. You can use combinations to find the number of outcomes in a sample space or in an event.

Your Turn

6. A math department has a large database of true-false questions, half of which are true and half of which are false, that are used to create future exams. A new test is created by randomly selecting 6 questions from the database. What is the probability the new test contains at least 2 questions where the correct answer is “true”?

The number of outcomes in the sample space \( S \) can be found by using the Fundamental Counting Principle since each question is either true or false.

Let \( A \) be the event that at most 2 questions are true. This is the sum of 3 events: 2 true questions, 1 true question, or no true questions.

\[
n(A) = \binom{6}{2} + \binom{6}{1} + \binom{6}{0} = \frac{6!}{2!4!} + \frac{6!}{1!5!} + \frac{6!}{0!6!} = 15 + 6 + 1 = 22
\]

Because the questions are equally likely to be true or false, the probability that the test contains at most 2 true questions is

\[
P(A) = \frac{n(A)}{n(S)} = \frac{22}{64} = \frac{11}{32}
\]

7. There are equally many boys and girls in the senior class. If 5 seniors are randomly selected to form the student council, what is the probability the council will contain at least 3 girls?

The number of outcomes in the sample space \( S \) can be found by using the Fundamental Counting Principle since each selection is either a boy or a girl.

Let \( A \) be the event that at least 3 girls are selected. This is the sum of 3 events: selecting 3 girls, 4 girls, or 5 girls.

\[
n(A) = \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = \frac{5!}{3!2!} + \frac{5!}{4!1!} + \frac{5!}{5!0!} = 10 + 5 + 1 = 16
\]

Because a senior is equally likely to be a boy or a girl, the probability that the council will contain at least 3 girls is

\[
P(A) = \frac{n(A)}{n(S)} = \frac{16}{32} = \frac{1}{2}
\]

Elaborate

8. Discussion A coin is flipped 5 times, and the result of heads or tails is recorded. To find the probability of getting tails at least once, the events of 1, 2, 3, 4, or 5 tails can be added together. Is there a faster way to calculate this probability?

The sum of the probabilities of all possible outcomes is equal to 1. Determine the probability of getting no tails (or 5 heads) and subtract this value from 1.

9. If \( \binom{a}{b} = \binom{b}{a} \), what is the relationship between \( a \) and \( b \)? Explain your answer.

The equation is true if \( \frac{n!}{a!(n-a)!} = \frac{n!}{b!(n-b)!} \). This will occur when \( a = b \) or \( a + b = n \).

10. Essential Question Check-In How do you determine whether choosing a group of objects involves combinations?

Combinations are used when the order of selection does not matter.

LANGUAGE SUPPORT

Connect Vocabulary

Have students make a chart to summarize what they know about combinations. Sample:

<table>
<thead>
<tr>
<th>Combination</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Definition</strong></td>
</tr>
<tr>
<td><strong>Formula</strong></td>
</tr>
<tr>
<td><strong>Example</strong></td>
</tr>
</tbody>
</table>
1. A cat has a litter of 6 kittens. You plan to adopt 2 of the kittens. In how many ways can you choose 2 of the kittens from the litter?

\[ nC_2 = \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!} = \frac{720}{48} = 15 \text{ ways} \]

2. An amusement park has 11 roller coasters. In how many ways can you choose 4 of the roller coasters to ride during your visit to the park?

\[ _{11}C_4 = \frac{11!}{4!(11-4)!} = \frac{11!}{4!7!} = \frac{39,916,800}{120,960} = 330 \text{ ways} \]

3. Four students from 30-member math club will be selected to organize a fundraiser. How many groups of 4 students are possible?

\[ _{30}C_4 = \frac{30!}{4!(30-4)!} = \frac{30!}{4!26!} = 27,405 \text{ groups} \]

4. A school has 5 Spanish teachers and 4 French teachers. The school’s principal randomly chooses 2 of the teachers to attend a conference. What is the probability that the principal chooses 2 Spanish teachers?

The sample space \( S \) consists of combinations of 2 teachers chosen from the 9 teachers, and event \( A \) consists of combinations of 2 teachers chosen from the 5 Spanish teachers.

\[ n(S) = _{9}C_2 = \frac{9!}{2!(9-2)!} = \frac{9!}{2!7!} = \frac{362,880}{10,080} = 36 \]

\[ n(A) = _{5}C_2 = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{120}{12} = 10 \]

\[ P(A) = \frac{n(A)}{n(S)} = \frac{10}{36} = \frac{5}{18} \]

5. There are 6 fiction books and 8 nonfiction books on a reading list. Your teacher randomly assigns you 4 books to read over the summer. What is the probability that you are assigned all nonfiction books?

The sample space \( S \) consists of combinations of 4 books chosen from the 14 books, and event \( A \) consists of combinations of 4 books chosen from the 8 nonfiction books.

\[ n(S) = _{14}C_4 = \frac{14!}{4!(14-4)!} = \frac{14!}{4!10!} = 1,001 \]

\[ n(A) = _{8}C_4 = \frac{8!}{4!(8-4)!} = \frac{8!}{4!4!} = 70 \]

\[ P(A) = \frac{n(A)}{n(S)} = \frac{70}{1,001} = \frac{10}{143} \]

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**Exercise 1–12**  
1. Recall of Information  
   **MP.2** Reasoning

2. Skills/Concepts  
   **MP.4** Modeling

3. Skills/Concepts  
   **MP.1** Problem Solving

4. Skills/Concepts  
   **MP.4** Modeling

5. Skills/Concepts  
   **MP.2** Reasoning

6. Strategic Thinking  
   **H.O.T.** MP.3 Logic
INTEGRATE MATHEMATICAL PRACTICES

Focus on Communication

MP.3 Ask students to share their rationales for solving probability problems involving combinations. In particular, have them explain how they computed the values for \( n \) and \( n(A) \) to find the probability.

AVOID COMMON ERRORS

Students often confuse permutations and combinations. They may not recognize which should be applied or they may apply the wrong formula. Have students begin by deciding whether the order is important or not. Then have them look up the formula. Note that this process may have to be repeated several times to solve a probability problem.

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6. A bag contains 26 tiles, each with a different letter of the alphabet written on it. You choose 3 tiles from the bag without looking. What is the probability that you choose the tiles with the letters A, B, and C?

Let \( S \) be the sample space, which consists of combinations of 3 tiles chosen from 26 tiles, and let \( A \) be the event that you choose the tiles with the letters A, B, and C.

\[
\begin{align*}
\text{n}(S) &= \binom{26}{3} = \frac{26!}{3!(26-3)!} = \frac{26!}{3!23!} = 2600 \\
\text{n}(A) &= \binom{3}{3} = \frac{3!}{3!(3-3)!} = \frac{3!}{3!0!} = 1 \rightarrow P(A) = \frac{n(A)}{n(S)} = \frac{1}{2600}
\end{align*}
\]

7. You are randomly assigned a password consisting of 6 different characters chosen from the digits 0 to 9 and the letters A to Z. As a percent, what is the probability that you are assigned a password consisting of only letters? Round your answer to the nearest tenth of a percent.

Let \( S \) be the sample space, which consists of combinations of 6 characters chosen from 36 characters, and let \( A \) be the event that you are assigned a password consisting of only letters.

\[
\begin{align*}
\text{n}(S) &= \binom{36}{6} = \frac{36!}{6!(36-6)!} = \frac{36!}{6!30!} = 1,947,792 \\
\text{n}(A) &= \binom{26}{6} = \frac{26!}{6!(26-6)!} = \frac{26!}{6!20!} = 230,230 \\
P(A) &= \frac{n(A)}{n(S)} = \frac{230,230}{1,947,792} \approx 11.8\%
\end{align*}
\]

8. A bouquet of 6 flowers is made up by randomly choosing between roses and carnations. What is the probability the bouquet will have at most 2 roses?

Let \( S \) be the sample space, which consists of combinations of 6 flowers (each of which is a rose or a carnation), and let \( A \) be the event that the bouquet will have at most 2 roses.

\[
\begin{align*}
\text{n}(S) &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64 \\
\text{n}(A) &= \binom{2}{2} + \binom{2}{1} + \binom{2}{0} = 1 + 2 + 1 = 22 \\
\text{Because a flower is equally likely to be a rose or a carnation, } P(A) &= \frac{n(A)}{n(S)} = \frac{22}{64} = \frac{11}{32}.
\end{align*}
\]

9. A bag of fruit contains 10 pieces of fruit, chosen randomly from bins of apples and oranges. What is the probability the bag contains at least 6 oranges?

Let \( S \) be the sample space, which consists of combinations of 3 tiles chosen from 26 tiles, and let \( A \) be the event that the bag contains at least 6 oranges.

\[
\begin{align*}
\text{n}(S) &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10} = 1024 \\
\text{n}(A) &= \binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10} = 210 + 120 + 45 + 10 + 1 = 386 \\
\text{Because a piece of fruit is equally likely to be an apple or an orange, } P(A) &= \frac{n(A)}{n(S)} = \frac{386}{1024} = \frac{193}{512}.
\end{align*}
\]
10. You flip a coin 10 times. What is the probability that you get at most 3 heads?

Let \( S \) be the sample space, which consists of the results of 10 coin flips (each of which is either heads or tails), and let \( A \) be the event that you get at most 3 heads.

\[
\begin{align*}
 n(S) &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10} = 1024 \\
 n(A) &= _{10}C_3 + _{10}C_2 + _{10}C_1 = 120 + 45 + 10 + 1 = 176
\end{align*}
\]

Because a flip is equally likely to result in heads or tails, \( P(A) = \frac{n(A)}{n(S)} = \frac{176}{1024} = \frac{11}{64}. \)

11. You flip a coin 8 times. What is the probability you will get at least 5 heads?

Let \( S \) be the sample space, which consists of the results of 8 coin flips (each of which is either heads or tails), and let \( A \) be the event that you get at least 5 heads.

\[
\begin{align*}
 n(S) &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^8 = 256 \\
 n(A) &= _{8}C_5 + _{8}C_6 + _{8}C_7 + _{8}C_8 = 56 + 28 + 8 + 1 = 93
\end{align*}
\]

Because a flip is equally likely to result in heads or tails, \( P(A) = \frac{n(A)}{n(S)} = \frac{93}{256}. \)

12. You flip a coin 5 times. What is the probability that every result will be tails?

Let \( S \) be the sample space, which consists of the results of 5 coin flips (each of which is either heads or tails), and let \( A \) be the event that every result will be tails.

\[
\begin{align*}
 n(S) &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32 \\
 n(A) &= _{5}C_5 = 1
\end{align*}
\]

Because a flip is equally likely to result in heads or tails, \( P(A) = \frac{n(A)}{n(S)} = \frac{1}{32}. \)

13. There are 12 balloons in a bag: 3 each of blue, green, red, and yellow.

Three balloons are chosen at random. Find the probability that all 3 balloons are green.

Let \( S \) be the sample space, which consists of combinations of the 3 balloons chosen from the 12 balloons, and let \( A \) be the event that all 3 balloons are green.

\[
\begin{align*}
 n(S) &= _{12}C_3 = \frac{12!}{3!(12 - 3)!} = 220 \\
 n(A) &= _{3}C_3 = 1
\end{align*}
\]

\[
P(A) = \frac{n(A)}{n(S)} = \frac{1}{220}
\]

14. There are 6 female and 3 male kittens at an adoption center. Four kittens are chosen at random. What is the probability that all 4 kittens are female?

Let \( S \) be the sample space, which consists of combinations of 4 kittens chosen from the 9 kittens, and let \( A \) be the event that all 4 kittens are female.

\[
\begin{align*}
 n(S) &= _{9}C_4 = \frac{9!}{4!(9 - 4)!} = \frac{9!}{4!5!} = 126 \\
 n(A) &= _{6}C_4 = \frac{6!}{4!(6 - 4)!} = \frac{6!}{4!2!} = 15
\end{align*}
\]

\[
P(A) = \frac{n(A)}{n(S)} = \frac{15}{126} = \frac{5}{42}
\]

PEER-TO-PEER

Have students work with a partner to write a probability problem about numbers, letters, or both in which a combination is needed to count the objects. Have students solve their problems. Then ask them to exchange problems with another pair to solve. Have the pairs review each other’s solution methods.

AVOID COMMON ERRORS

Students sometimes attempt to simplify permutations or combinations by canceling factors. Remind students of the meaning of factorials, and suggest that they write out the multiplication to determine which factors actually cancel.
There are 21 students in your class. The teacher wants to send 4 students to the library each day. The teacher will choose the students to go to the library at random each day for the first four days from the list of students who have not already gone. Answer each question.

15. What is the probability you will be chosen to go on the first day?

Let \( S \) be the sample space, which consists of the combinations of 4 students chosen from the 21 students, and let \( A \) be the event that you will be chosen to go on the first day.

\[
\begin{align*}
\text{n}(S) &= \binom{21}{4} = \frac{21!}{4!(21-4)!} = \frac{21!}{4!17!} = 5985 \\
\text{n}(A) &= \binom{20}{3} = \frac{20!}{3!(20-3)!} = \frac{20!}{3!17!} = 1140 \\
\text{P}(A) &= \frac{\text{n}(A)}{\text{n}(S)} = \frac{1140}{5985} = \frac{4}{21}
\end{align*}
\]

16. If you have not yet been chosen to go on days 1–3, what is the probability you will be chosen to go on the fourth day?

12 students have already gone, leaving 9 students to go to the library.

Let \( S \) be the sample space, which consists of the combinations of 4 students chosen from the 9 students, and let \( A \) be the event that you will be chosen to go on the fourth day.

\[
\begin{align*}
\text{n}(S) &= \binom{9}{4} = \frac{9!}{4!(9-4)!} = \frac{9!}{4!5!} = 126 \\
\text{n}(A) &= \binom{8}{3} = \frac{8!}{3!(8-3)!} = \frac{8!}{3!5!} = 56 \\
\text{P}(A) &= \frac{\text{n}(A)}{\text{n}(S)} = \frac{56}{126} = \frac{4}{9}
\end{align*}
\]

17. Your teacher chooses 2 students at random to represent your homeroom. The homeroom has a total of 30 students, including your best friend. What is the probability that you and your best friend are chosen?

Let \( S \) be the sample space, which consists of the combinations of 2 students chosen from the 30 students, and let \( A \) be the event that you and your best friend are chosen.

\[
\begin{align*}
\text{n}(S) &= \binom{30}{2} = 435 \\
\text{n}(A) &= \binom{29}{1} = 29 \\
\text{P}(A) &= \frac{\text{n}(A)}{\text{n}(S)} = \frac{29}{435}
\end{align*}
\]
There are 12 peaches and 8 bananas in a fruit basket. You get a snack for yourself and three of your friends by choosing four of the pieces of fruit at random. Answer each question.

18. What is the probability that all 4 are peaches?

The sample space $S$ consists of the combinations of 4 pieces of fruit from the 20 pieces of fruit, and event $A$ consists of combinations of 4 peaches.

$$n(S) = \binom{20}{4} = \frac{20!}{4!(20-4)!} = 4845$$

$$n(A) = \binom{12}{4} = \frac{12!}{4!(12-4)!} = 495$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{495}{4845} = \frac{33}{323}$$

19. What is the probability that all 4 are bananas?

The sample space $S$ consists of the combinations of 4 pieces of fruit from the 20 pieces of fruit, and event $A$ consists of combinations of 4 bananas.

$$n(S) = \binom{20}{4} = \frac{20!}{4!(20-4)!} = 4845$$

$$n(A) = \binom{8}{4} = \frac{8!}{4!(8-4)!} = 70$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{70}{4845} = \frac{14}{969}$$

20. There are 30 students in your class. Your science teacher will choose 5 students at random to create a group to do a project. Find the probability that you and your 2 best friends in the science class will be chosen to be in the group.

Since 3 of the group members are you and your friends, the additional 2 group members can come from any combination of the students left in class.

The sample space $S$ consists of the combinations of 5 students from the 30 students, and event $A$ consists of combinations of 2 students from the 27 who are left.

$$P(A) = \frac{n(A)}{n(S)} = \frac{\binom{27}{2}}{\binom{30}{5}} = \frac{351}{142,506} = \frac{1}{406}$$

21. On a television game show, 9 members of the studio audience are randomly selected to be eligible contestants.

a. Six of the 9 eligible contestants are randomly chosen to play a game on the stage. How many combinations of 6 players from the group of eligible contestants are possible?

$$\binom{9}{6} = \frac{9!}{6!(9-6)!} = \frac{9!}{6!3!} = 84$$

b. You and your two friends are part of the group of 9 eligible contestants. What is the probability that all three of you are chosen to play the game on stage? Explain how you found your answer.

The sample space $S$ consists of the combinations of 6 contestants from the 9 who are eligible.

$$n(S) = \binom{9}{6} = 84$$

After you and your friends are chosen, 3 other contestants from the remaining 6 can be chosen in any combination, so event $A$ consists of combinations of 3 contestants from the 6 who are left.

$$n(A) = \binom{6}{3} = 20, P(A) = \frac{n(A)}{n(S)} = \frac{20}{84} = \frac{5}{21}$$
22. Determine whether you should use permutations or combinations to find the number of possibilities in each of the following situations. Select the correct answer for each lettered part.

a. Selecting a group of 5 people from a group of 8 people
   - permutation
   - combination

b. Finding the number of combinations for a combination lock
   - permutation
   - combination

c. Awarding first and second place ribbons in a contest
   - permutation
   - combination

d. Choosing 3 books to read in any order from a list of 7 books
   - permutation
   - combination

a. It doesn’t matter in what order the people are selected in.

b. Order matters: numbers have to be in a specific order to open the lock.

c. Order matters: awarding Sam first place and Elena second is different from awarding Elena first place and Sam second.

d. It doesn’t matter in what order the books are chosen.

H.O.T. Focus on Higher Order Thinking

23. Communicate Mathematical Ideas Using the letters A, B, and C, explain the difference between a permutation and a combination.

In permutations, order matters. In combinations, order does not matter. In a permutation of A, B, and C, ABC is different from CBA, so they would be counted as two different permutations. In a combination, ABC is the same as CBA, and would not be counted again.

24. a. Draw Conclusions Calculate \(_{6}C_{4}\) and \(_{4}C_{4}\).

\[
_{6}C_{4} = \frac{10!}{6!(10 - 6)!} = \frac{10!}{6!4!} = \frac{3,628,800}{17,280} = 210
\]

\[
_{4}C_{4} = \frac{10!}{4!(10 - 4)!} = \frac{10!}{4!6!} = \frac{3,628,800}{17,280} = 210
\]

b. What do you notice about these values? Explain why this makes sense.

\(_{6}C_{4} = \(_{4}C_{4}\) = 210; it makes sense that these values are equal because every combination of 6 objects that are selected has a corresponding combination of 4 objects that are not selected.

c. Use your observations to help you state a generalization about combinations.

In general, \(_{n}C_{r} = \(_{n}C_{n-r} \).
25. Justify Reasoning  Use the formula for combinations to make a generalization about \( \binom{n}{r} \). Explain why this makes sense.

Using the formula for combinations and the fact that

\[ 0! = 1, \binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1; \text{ this makes sense because there is only 1 combination of 0 objects taken 0 at a time.} \]

26. Explain the Error  Describe and correct the error in evaluating \( \binom{n}{r} \).

\[ \binom{9}{5} = \frac{9!}{(5-4)!} = \frac{9!}{1!} = 9 \cdot \frac{8!}{8!} = 9 \cdot 8 = 72 \]

The answer given was \( \binom{4}{5} \) not \( \binom{4}{4} \).

Lesson Performance Task

1. In the 2012 elections, there were six candidates for the United States Senate in Vermont. In how many different orders, from first through sixth, could the candidates have finished?

2. The winner of the Vermont Senatorial election received 208,253 votes, 71.1% of the total votes cast. The candidate coming in second received 24.8% of the vote. How many votes did the second-place candidate receive? Round to the nearest ten.

3. Following the 2012 election there were 53 Democratic, 45 Republican, and 2 Independent senators in Congress.
   a. How many committees of 5 Democratic senators could be formed?
   b. How many committees of 48 Democratic senators could be formed?
   c. Explain how a clever person who knew nothing about combinations could guess the answer to (b) if the person knew the answer to (a).

4. Following the election, a newspaper printed a circle graph showing the make-up of the Senate. How many degrees were allotted to the sector representing Democrats, how many to Republicans, and how many to Independents?

   a. Term 0
   b. Term 1
   c. Term 2
   d. Term 3
   e. Term 4
   f. Term 5
   g. Term 6

Find \( \binom{3}{2} \), \( \binom{5}{3} \), and \( \binom{6}{4} \). Then propose a connection between the terms in the triangle and the quantity \( \binom{n}{m} \). (You’ll find the connection easiest to spot by numbering the first term in each row Term 0. So, Term 3 in Row 6 is 20.) \( \binom{n}{m} \) equals Term \( m \) in Row \( n \).

What is the 14th term in Row 17 of Pascal’s Triangle? \( \binom{17}{14} = 680 \)

EXTENSION ACTIVITY

Each day, Senator Smith leaves his office and walks to the Committee Room along a grid of hallways that forms a 5 by 5 square. He moves only right (R) and down (D). The path shown can be written RRRDDDRDDDR. The senator has developed a method to use combinations to find the number of different ways he can complete his walk. What is the method? How many ways can he do it? Find all possible combinations of five R’s and five D’s. \( \frac{10!}{5!5!} = 252 \) ways.